PLANE
and
SPHERICAL
Trigonometry

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WITH TABLES
TO FOUR PLACES OF DECIMALS

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PREFACE

THIS BOOK is a complete rewriting of the author's *Elements of Trigonometry.* The direct approach to the various topics has been maintained, but the explanations have been amplified and much more use is made of illustrative worked examples. Great care has been taken to make these examples instructive and to serve as patterns for the problem work of the student. Many elementary exercises and problems of current interest have been added.

The drill problems and the applications cover a sufficient range to give the student in technical courses a working knowledge of trigonometry as a tool subject.

Among the applications of plane trigonometry those relating to mensuration have received full treatment, as also the subjects of vectors, plane surveying and plane sailing. A treatment of the mil unit of angle and its applications is given along with a brief table of the functions at intervals of 40 mils.

The ideas of inverse functions and of trigonometric equations are introduced early and later amplified in a separate chapter.

The subject of spherical trigonometry is treated in two chapters. In the first of these the formulas are derived and applied to the solution of spherical triangles; the second is devoted to applications, principally in navigation and nautical astronomy. Considerable attention is given to the use of the haversine and a four-place table is provided so that the student may become familiar with the use of this important function.

The subject of great circle sailing, including the "vertex method," the construction and use of the Mercator chart, and the basic problems of nautical astronomy have received careful attention.

For a brief course, or where more time is desired for spherical trigonometry, the following curtailments and omissions are advised.

1. Omit the long list of identities of §77.
2. Take only a limited selection of the problems of §90.
3. Omit Chapter IX, on inverse functions and trigonometric

equations. The earlier treatment of these subjects is sufficient for a brief course.

4. Omit Chapter X, on analytical trigonometry.

With these omissions the presentation of the subject is suitable for use in the senior high school.

The author wishes to acknowledge his indebtedness to Professor C. J. Rees of the University of Delaware and Professor R. H. Marquis of Ohio University who have read the manuscript and offered valuable suggestions.

W. C. BRENKE

August, 1942

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CHAPTER

I

THE TRIGONOMETRIC FUNCTIONS

1. Why we study trigonometry.

The subject of trigonometry may be considered, in one of its two principal aspects, to round out the subject of geometry. It supplies the means of expressing in an exact quantitative way much that geometry does only qualitatively. Here are some examples.

(1) Geometry tells us that, in a given circle, a given central angle subtends a definite chord, and how to construct the figure on any desired scale. Trigonometry enables us to state an exact formula for the length of the chord. The great astronomer Ptolemy calculated a table of chords corresponding to various central angles.

(2) Geometry tells us that a triangle is completely determined when one side and two angles are given, and how to construct the triangle on any desired scale. Trigonometry provides us with exact formulas for calculating the unknown parts of the triangle.

(3) Geometry tells us how to construct the resultant force of two given forces. Trigonometry enables us to calculate this resultant force.

(4) The geometry of the sphere, rounded out by spherical trigonometry, is of basic importance to the navigator and astronomer.

The second major aspect of our subject results from the fact that the "trigonometric functions", which we shall study presently, are peculiarly adapted to express many important relations in physics and mechanics and related fields. These functions are among the most useful and basic tools which are employed in the application of mathematics to the physical sciences.

To indicate at least one such field of applications we note that the studies of periodic phenomena, such as the vibration of a pendulum or of a violin string, the periodic motion of a planet about the sun or of an electron about the nucleus of its atom, and innumerable other events of a regularly recurring character, have their roots in the study of trigonometry.

2. Angles of any magnitude, positive or negative.

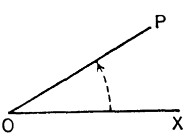


FIG. 1

Consider $\angle XOP$ (figure) as generated by a moving line which rotates about O from the position OX to the position OP.

Divide the plane into four *quadrants* (I, II, III, and IV in the figure below) by means of two *rectangular axes* $X'X$ and $Y'Y$.

Quadrant I is that covered by a half-line or ray rotating from

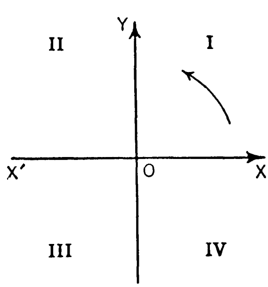


FIG. 2a

OX to OY in the direction of the curved arrow, counterclockwise, the angle turned through being 90°. Let a moving ray start from the position OX, Fig. 2b, and rotate into the positions

OP_1, OP_2, OP_3, and OP_4 successively, thus generating the angles XOP_1, XOP_2, XOP_3, and XOP_4 respectively.

OX is called the ***initial line,*** and OP_1 the ***terminal line*** of the

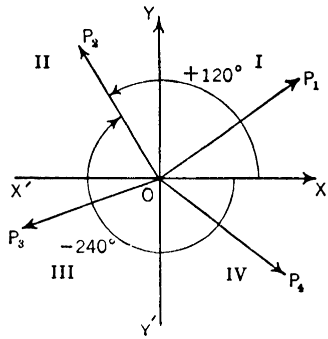


FIG. 2b

angle XOP_1, and similarly for any other angle.

An angle is ***positive*** when the generating ray rotates ***counter-clockwise*** (in the direction of the curved arrow in the figure), ***negative*** when the generating ray moves ***clockwise.***

The ***quadrant of an angle*** is that quadrant in which its terminal line lies. The angle is said to lie in this quadrant.

The initial line OX, and any terminal line, as OP_2, may always be considered to form two angles numerically less than 360°, as + 120° and − 240° in the figure.

When the moving ray rotates from OX through more than one complete revolution, an angle greater than 360° is generated. Thus a rotation in the positive direction (positive rotation) through $1\frac{1}{3}$ revolutions generates an angle of 480°, lying in the second quadrant; a negative rotation through $2\frac{1}{6}$ revolutions generates an angle of − 780°, lying in the fourth quadrant.

3. Rectangular coordinates.

With respect to the reference frame of Fig. 2a, any point in the plane may be located by means of its distances from the two reference lines and by adopting a rule to distinguish between the different quadrants.

The two distances of point P from the reference lines are

usually indicated by letters, as x and y in Fig. 3, and are named as follows:

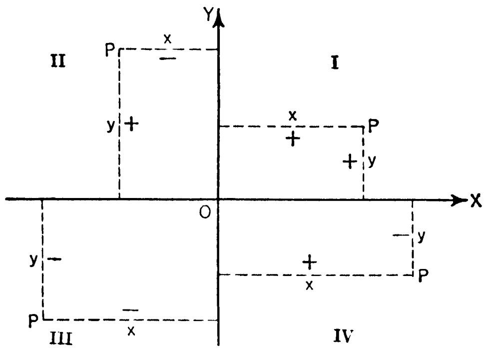


Fig. 3

$x =$ *abscissa* of point P, $y =$ *ordinate* of point P.

The number pair (x, y) *are called the rectangular coordinates of point* P.

To distinguish between the quadrants we use signed numbers for the values of x and y, as indicated in the figure. This may be summed up in the following table.

Quadrant	Abscissa	Ordinate
I	+	+
II	−	+
III	−	−
IV	+	−

Exercise. On cross-ruled paper draw a pair of reference lines, mark them with arrows to indicate the first quadrant, and locate the points (x, y) determined by the following pairs of numbers. The first number is the abscissa, the second the ordinate.

(2, 3), (4, 2), (− 2, 3), (− 4, 2), (− 2, − 3), (− 4, − 2), (2, − 3), (4, − 2), (5, 0), (0, 3), (− 5, 0), (0, − 3).

If P denotes any of these points estimate as well as you can the number of degrees in the positive angle XOP; in the negative angle XOP.

4. The trigonometric functions of any angle.

In Fig. 3 draw a line from O through P, where P may lie in any of the four quadrants. Consider OP as the terminal line of

an angle with OX as the initial line. We are thus led to Fig. 4, in which x is the abscissa and y the ordinate of point P, and $r = OP$ is the distance of point P from the origin. The distance OP is always considered to be positive, so that r always stands for a positive number.

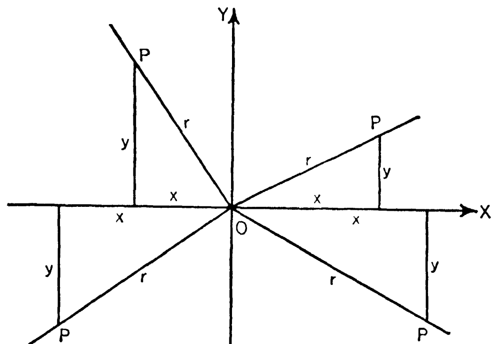


FIG. 4

By taking the numbers x, y, r in pairs we can form *six ratios*, namely

$$\frac{y}{r}, \quad \frac{x}{r}, \quad \frac{y}{x}, \quad \frac{r}{y}, \quad \frac{r}{x}, \quad \frac{x}{y}.$$

These ratios are defined to be the *six trigonometric functions* of angle XOP, and are named as follows.

The sine of angle XOP $= \dfrac{\textbf{ordinate (of } P)}{\textbf{distance (of } P)}$.

The cosine of angle XOP $= \dfrac{\textbf{abscissa}}{\textbf{distance}}$.

The tangent of angle XOP $= \dfrac{\textbf{ordinate}}{\textbf{abscissa}}$.

The cotangent of angle XOP $= \dfrac{\textbf{abscissa}}{\textbf{ordinate}}$.

The secant of angle XOP $= \dfrac{\textbf{distance}}{\textbf{abscissa}}$.

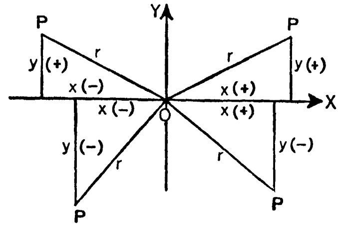
The cosecant of angle XOP $= \dfrac{\textbf{distance}}{\textbf{ordinate}}$.

It will be convenient to use a single letter to designate our angle XOP. For this purpose we shall use the Greek letter alpha, and put α = angle XOP. Introducing also the letters x, y and r, and abbreviating the names of the trigonometric functions, we have

$$\sin\alpha = \frac{y}{r};\quad \csc\alpha = \frac{r}{y};$$

$$\cos\alpha = \frac{x}{r};\quad \sec\alpha = \frac{r}{x};$$

$$\tan\alpha = \frac{y}{x};\quad \cot\alpha = \frac{x}{y}.$$



Here x and y stand for signed numbers according to the quadrant of the angle α; the distance r is always taken as a positive number.

What is the effect of changing the position of P along the terminal side of the angle? The values of x, y and r will change, but, because of the similarity of the triangles, their ratios will remain unchanged. Hence the trigonometric functions depend only on the angle α, and not at all on the particular point P which we select on the terminal side of the angle.

The signs of the trigonometric functions.

According to the definitions we can construct a table showing the signs of the trigonometric functions in the various quadrants. In quadrant I, x, y, r all are positive and likewise the ratio of any pair of them is positive. Therefore, in quadrant I all the six trigonometric functions are positive.

In quadrant II, y and r are positive and x is negative. Therefore the sine function (ratio y/r) and the cosecant function (ratio r/y) are positive; the other four functions are negative.

Table of signs of the trigonometric functions

Quadr.	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\csc\alpha$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Let the student verify carefully the signs in this table. He should be prepared to state instantly the sign of any function in any quadrant.

Observe that in the first quadrant all the functions are positive; in the other quadrants a function and its reciprocal are positive, the remaining four are negative.

EXERCISES 1.

Determine the values of the six trigonometric functions of angle XOP when the coordinates (x, y) of P are as given below. Give exact values.

1. (3, 4).	**5.** (12, 5).	**9.** (8, 15).	**13.** (2, 3).
2. (– 3, 4).	**6.** (– 12, 5).	**10.** (– 8, 15).	**14.** (– 2, 3).
3. (3, – 4).	**7.** (12, – 5).	**11.** (8, – 15).	**15.** (2, – 3).
4. (– 3, – 4).	**8.** (– 12, – 5).	**12.** (– 8, – 15).	**16.** (– 2, – 3).

5. Approximate values of the functions of any angle.

If in the last figure the distances OP had been taken all of the same length, all the points P would lie on the circumference of a circle with center at O.

Let us draw a circle with O as center and unit radius (figure; 1 = 10 small divisions). Then for any angle XOP we have

$$\sin XOP = \frac{MP}{1} = MP,$$

$$\cos XOP = \frac{OM}{1} = OM.$$

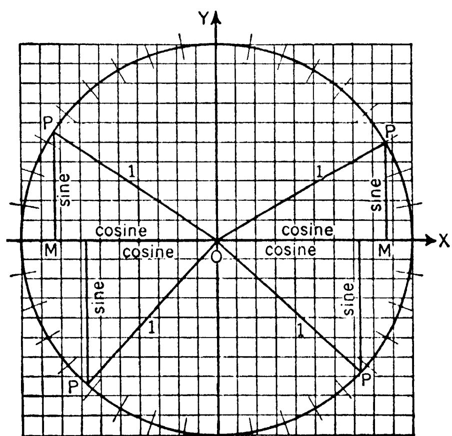


FIG. 5

Thus the figure shows

$$\begin{array}{llll}
\sin 30° = .5 & \text{and} & \cos 30° = .86; \\
\sin 147° = .56 & \text{and} & \cos 147° = -.83; \\
\sin 228° = -.73 & \text{and} & \cos 228° = -.67; \\
\sin 317° = -.69 & \text{and} & \cos 317° = .72.
\end{array}$$

By noting the values of MP at regular intervals as P moves around the circumference, a complete table of values may be constructed.

Hence approximate values of the sines and cosines of all angles may be read off directly from the figure. The other functions may be obtained by division, since $\tan XOP = \frac{MP}{OM}$, etc. They may also be constructed graphically by a method explained in the next article.

Exercise. By use of the figure determine to two decimal places the values needed to fill out the following table.

Angle α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\csc\alpha$
30°						
45°						
60°						
120°						
135°						
150°						
210°						
225°						
240°						
300°						
315°						
330°						

6. Line values of the trigonometric functions.

The lines MP and OM, Fig. 5, *measured with OP as a unit of length,* represent the values of $\sin XOP$ and $\cos XOP$ respectively. They are called the *line values* of these functions.

The origin of the term *sine* is obscure. The Hindus used *jya*

meaning *chord* and an Arabic distortion of the Hindu word was rendered by *sinus* in later Latin works.

We shall note briefly the line values of the other trigonometric functions for the case of acute angles. Other angles may be treated similarly, with suitable consideration of signs according to the quadrant.

In Fig. 6a let α be an acute angle with initial line OX and terminal line OQ, NQ being tangent to the circle of radius 1 and center at the vertex of the angle. In triangle ONQ:

$$\tan \alpha = \frac{\text{ord. } NQ}{\text{abs. } ON} = \frac{NQ}{1} = NQ.$$

$$\sec \alpha = \frac{\text{dist. } OQ}{\text{abs. } ON} = \frac{OQ}{1} = OQ.$$

Hence $\tan \alpha$ is measured by a segment of a line tangent to the circle and $\sec \alpha$ is measured by a segment of a secant line. This indicates the origin of the names of these functions.

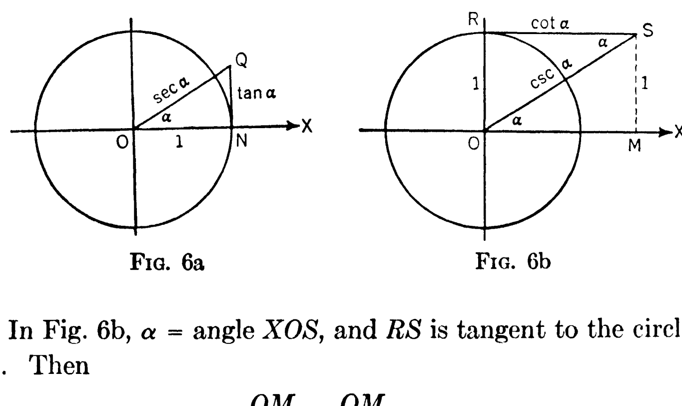


FIG. 6a FIG. 6b

In Fig. 6b, α = angle XOS, and RS is tangent to the circle at R. Then

$$\cot \alpha = \frac{OM}{MS} = \frac{OM}{1} = OM = RS.$$

$$\csc \alpha = \frac{OS}{MS} = \frac{OS}{1} = OS.$$

NOTE. If, in Fig. 6a, we produce line NQ upward indefinitely and let angle α increase toward 90°, we see that both $\tan \alpha$ and $\sec \alpha$ will increase very rapidly and without limit. Fig. 6b shows similarly that $\cot \alpha$ and $\csc \alpha$ increase rapidly and without limit as angle α diminishes toward 0°.

7. The trigonometric functions of acute angles.

Let us consider a given acute angle α and construct a right triangle ABC (Fig. 7a) containing this acute angle. Place the $\triangle ABC$ in our reference frame, (Fig. 2a), so that the vertex A of angle α shall fall at O, AC shall fall along the initial line OX, and AB shall fall in the first quadrant. (Fig. 7b).

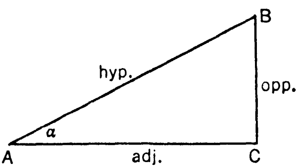


FIG. 7a

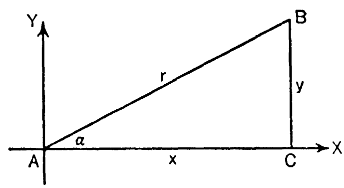


FIG. 7b

Using point B on the terminal side of angle α as point P in Fig. 4 we shall have

$$\begin{aligned} AC &= x = \text{abscissa of point } B; \\ CB &= y = \text{ordinate of point } B; \\ AB &= r = \text{distance of point } B. \end{aligned}$$

We can then write down the six trigonometric functions of angle α according to the definitions. For example,

$$\sin \alpha = \frac{\text{ordinate of } B}{\text{distance of } B} = \frac{CB}{AB} = \frac{y}{r}.$$

But in the original triangle ABC (Fig. 7a), CB is the side opposite angle α, and AB is the hypotenuse and therefore, with respect to the original triangle, we can say that the *sine of angle α is the ratio of the side opposite angle α to the hypotenuse.*

Any other right triangle containing the same acute angle α would be similar to $\triangle ABC$ and would have the ratio of any two of its sides equal to the ratio of the corresponding sides of $\triangle ABC$. Hence it would furnish the same values for the trigonometric functions of the angle α.

We may therefore restate our definitions of the six trigonometric functions, as applied to *acute angles.*

Let α be an acute angle, $\triangle ABC$ a right triangle containing

this angle, AB its hypotenuse, AC the side adjacent to $\angle\ \alpha$ and CB the side opposite to $\angle\ \alpha$. (Fig. 7a). Then

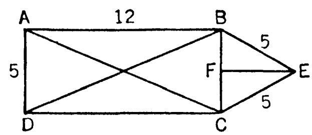
$$\sin\ \alpha = \frac{\text{opposite side}}{\text{hypotenuse}}. \qquad \csc\ \alpha = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

$$\cos\ \alpha = \frac{\text{adjacent side}}{\text{hypotenuse}}. \qquad \sec\ \alpha = \frac{\text{hypotenuse}}{\text{adjacent side}}.$$

$$\tan\ \alpha = \frac{\text{opposite side}}{\text{adjacent side}}. \qquad \cot\ \alpha = \frac{\text{adjacent side}}{\text{opposite side}}.$$

Exercise 1. Use Fig. 5 to obtain approximate values, to two decimal places, of the functions of 20°, 50° and 70°. Check by the table in §8.

Exercise 2. In the adjacent figure determine the exact values of the six functions of angle BAC; of angle CAD; of angle BEF, F being the midpoint of BC.



8. Brief table of the trigonometric functions.

Angle	Sin	Cos	Tan	Cot	Sec	Csc
0°						
5	0.087	0.996	0.087	11.430	1.004	11.474
10	0.174	0.985	0.176	5.671	1.015	5.759
15	0.259	0.966	0.268	3.732	1.035	3.864
20	0.342	0.940	0.364	2.747	1.064	2.924
25	0.423	0.906	0.466	2.145	1.103	2.366
30	0.500	0.866	0.577	1.732	1.155	2.000
35	0.574	0.819	0.700	1.428	1.221	1.743
40	0.643	0.766	0.839	1.192	1.305	1.556
45	0.707	0.707	1.000	1.000	1.414	1.414
50	0.766	0.643	1.192	0.839	1.556	1.305
55	0.819	0.574	1.428	0.700	1.743	1.221
60	0.866	0.500	1.732	0.577	2.000	1.155
65	0.906	0.423	2.145	0.466	2.366	1.103
70	0.940	0.342	2.747	0.364	2.924	1.064
75	0.966	0.259	3.732	0.268	3.864	1.035
80	0.985	0.174	5.671	0.176	5.759	1.015
85	0.996	0.087	11.430	0.087	11.474	1.004
90						

EXERCISES 2.

1. Obtain from this table the values of the six functions of 32.5° to three decimal places, assuming that they lie halfway between the values for 30° and 35°.

2. Obtain to three decimal places the values sin 31°, sin 32°, sin 33° and sin 34°, by breaking up the interval between sin 30° and sin 35° into five equal parts.

3. Obtain the values of cos 61°, cos 62°, cos 63°, cos 64°, and of sec 61°, sec 62°, sec 63°, sec 64°, to three decimal places.

4. Determine the angle α to the nearest degree if $\sin \alpha = 0.594$; if $\cos \alpha = 0.594$; if $\tan \alpha = 0.384$; if $\csc \alpha = 1.116$.

5. For what angle does $\sin \alpha = \cos \alpha$? $\tan \alpha = \cot \alpha$? $\sec \alpha = \csc \alpha$?

The following equations are exact; show that they are very nearly satisfied by the values taken from the table.

6. $2 \sin 30° \cos 30° = \sin 60°$.

7. $\cos^2 30° + \sin^2 30° = 1$.

8. $\cos^2 30° - \sin^2 30° = \cos 60°$.

9. $\cos^2 40° + \sin^2 40° = 1$.

10. $\cos^2 40° - \sin^2 40° = \cos 80°$.

9. The functions of 45°, 30° and 60°.

Any isosceles right triangle has each acute angle equal to 45°. A 30°–60° right triangle may be obtained by bisecting an equilateral triangle. The simplest numbers to use for the lengths of the sides are shown in Figs. 8a, 8b.

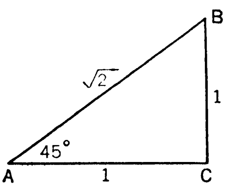


FIG. 8a

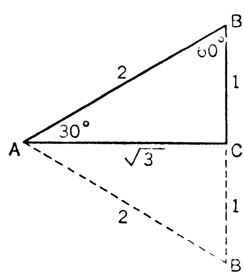


FIG. 8b

To obtain the functions of 45°, we apply the definitions of §7 to Fig. 8a.

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.707+. \qquad \csc 45° = \sqrt{2} = 1.414+.$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.707+. \qquad \sec 45° = \sqrt{2} = 1.414+.$$

$$\tan 45° = 1. \qquad \cot 45° = 1.$$

To obtain the functions of 30° and of 60° we apply our definitions to Fig. 8b. Note that side CB is opposite to 30° and also adjacent to 60°; AC is adjacent to 30° and opposite to 60°. The same triangle serves for both angles.

$$\sin 30^\circ = \cos 60^\circ = \tfrac{1}{2} = 0.5.$$

$$\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2} = 0.866+.$$

$$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.577+.$$

$$\csc 30^\circ = \sec 60^\circ = 2.$$

$$\sec 30^\circ = \csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.155+.$$

$$\cot 30^\circ = \tan 60^\circ = \sqrt{3} = 1.732+.$$

The functions of 30°, 45°, and 60° are so useful that the student should learn to read them off promptly from a mental picture of the isosceles right triangle and the bisected equilateral triangle.

EXERCISES

Verify the following equations by substituting the (exact) values of the functions.

1. $\sin 30^\circ \cot 30^\circ = \cos 30^\circ$.

2. $\tan 45^\circ \cos 45^\circ = \sin 45^\circ$.

3. $\sin 60^\circ \sec 60^\circ = \tan 60^\circ$.

4. $\cot 30^\circ \sec 30^\circ = \csc 30^\circ$.

5. $\cos 30^\circ \sec 60^\circ = \cot 30^\circ$.

6. $\sec 45^\circ \csc 45^\circ - \cot 45^\circ = \tan 45^\circ$.

7. $\cot 30^\circ \sin 60^\circ + \cos 60^\circ = \csc 30^\circ$.

8. $\tan 30^\circ + \tan 45^\circ = \tan 30^\circ(1 + \cot 30^\circ)$.

9. $(1 - \cos 45^\circ)(1 + \csc 45^\circ) = \sin 45^\circ$.

10. $(\csc 60^\circ + \cot 60^\circ)(\csc 60^\circ - \cot 60^\circ) = 1$.

10. Given one function, to determine the other functions.

When a function of an acute angle is given, the angle may be constructed by writing the given function as a fraction, and constructing a right triangle, two of whose sides are the numerator and denominator of this fraction respectively, or like multiples of these quantities. Also, since the third side of the triangle can be calculated from the other two, all the other functions of the angle may be found when one function is given.

Example 1.

$$\tan \alpha = \frac{3}{4}\left(= \frac{opp.\ side}{adj.\ side}\right).$$

Lay off $AC = 4$ and $CB = 3$, CB perpendicular to AC.

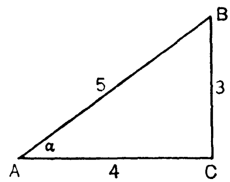


FIG. 9

Then $\qquad AB = \sqrt{4^2 + 3^2} = 5.$

Hence $\qquad \sin \alpha = \frac{3}{5};\ \cos \alpha = \frac{4}{5};$

$$\csc \alpha = \frac{5}{3};\ \sec \alpha = \frac{5}{4};\ \cot \alpha = \frac{4}{3}.$$

Scaling off the angle with a protractor, we have $\alpha = 37°$. By taking from the table the angle whose tangent is 0.75 we have $\alpha = 37°$ as before.

Example 2.

$$\sec \alpha = 3 \qquad \left(= \frac{3}{1} = \frac{hyp.}{adj.\ side}\right).$$

Lay off $AC = 1$. With A as center and radius $= 3$, strike an arc to cut the perpendicular drawn to AC at C. This determines the point B.

The solution may now be completed as in example 1.

Another method of constructing the triangle in this example is to calculate CB first, and then to proceed as in example 1.

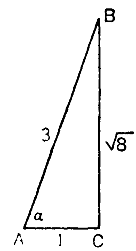


FIG. 10

11. EXERCISES 3

Determine the angle (approximately) and the remaining functions, when

1. $\sin \alpha = \frac{12}{13}$.	**6.** $\tan \alpha = \frac{5}{3}$.	**11.** $\sec \alpha = 2$.
2. $\sin \alpha = \frac{2}{5}$.	**7.** $\tan \alpha = 3$.	**12.** $\csc \alpha = \frac{3}{2}$.
3. $\sin \alpha = 0.4$.	**8.** $\tan \alpha = \sqrt{3}$.	**13.** $\cos \alpha = 0.3$.
4. $\cos \alpha = \frac{2}{5}$.	**9.** $\cot \alpha = 1$.	**14.** $\csc \alpha = 2.5$.
5. $\cos \alpha = \frac{1}{3}$.	**10.** $\cot \alpha = 2.5$.	**15.** $\tan \alpha = 10$.

16. Show that the equation $\sin \alpha = 2$ is impossible.

17. Show that the equation $\cos \alpha = 1.1$ is impossible.

18. Show that the equation $\sec \alpha = \frac{1}{2}$ is impossible.

19. Show that the equation $\csc \alpha = 0.9$ is impossible.

When α is an acute angle show that,

20. $\sin \alpha$ lies between 0 and 1.

21. $\cos \alpha$ lies between 0 and 1.

22. $\sec \alpha$ and $\csc \alpha$ are always greater than 1.

23. $\tan \alpha$ and $\cot \alpha$ may have any value from 0 to ∞.

12. Functions of complementary angles.

Since the sum of the two acute angles of a right triangle is 90°, they are complementary.

By definition we have, from Fig. 11,

$$\sin \beta = \frac{\text{opp. side}}{\text{hyp.}} = \frac{b}{c} = \cos \alpha.$$

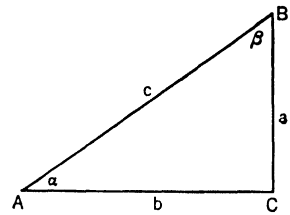


FIG. 11

By considering the other functions and tabulating results we have:

$$\begin{array}{lll} \sin \beta = \cos \alpha; & \tan \beta = \cot \alpha; & \csc \beta = \sec \alpha; \\ \cos \beta = \sin \alpha; & \cot \beta = \tan \alpha; & \sec \beta = \csc \alpha. \end{array}$$

Complementary functions, or cofunctions.

The cosine is called the complementary function to the sine and conversely. Similarly tangent and cotangent are mutually complementary, and secant and cosecant. The function which is complementary to a function is called its **cofunction.**

RULE: *Any function of an acute angle is equal to the cofunction of the complementary angle.*

Exercise. Verify this rule when $\alpha = 30°$, 45°, and 60°. See also the table of §8.

13. Application of the trigonometric functions to the solution of right triangles.

When two parts of a right triangle are known, exclusive of the right angle, the triangle may be constructed and the remaining parts determined graphically. By the aid of tables of the trigonometric functions, the unknown parts may also be calculated.

RULE: *When two parts of a right triangle are given (the right angle excepted) and a third part is required, write down that equation of §7 which involves the two given parts and the required part. Substitute in it the values of the given parts, and solve for the required part.*

An exceptional case arises when two sides are given and the third side is required. In this case we may use the formula $a^2 + b^2 = c^2$. It will usually be better, however, unless the given

sides are represented by simple numbers, to solve for one of the angles first, and then to obtain the third side from this angle and one of the given sides.

Example.

In right $\triangle ABC$, given angle $ACB = 90°$, angle $CAB = \alpha = 40°$, and side $b = 60$. Find the other parts of the triangle, c, a, and angle $ABC = \beta$.

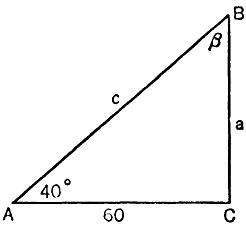


FIG. 12

To get β, we have $\beta = 90° - \alpha = 50°$.

To get a, take $\dfrac{a}{b} = \tan\alpha$ or $a = b\tan\alpha$.

Finally, c is determined from

$$\frac{b}{c} = \cos\alpha \quad \text{or} \quad c = \frac{b}{\cos\alpha} = b\sec\alpha.$$

From the table of §8, $\tan 40° = 0.839$

and $\sec 40° = 1.305$.

Hence $a = 60 \times 0.839 = 50.340$

and $c = 60 \times 1.305 = 78.300$.

As a check, we should have $a = c\cos\beta$, or $50.340 = 78.300 \times 0.643$.

14. EXERCISES 4

Determine the unknown parts of right triangle ABC, C being 90°, from the parts given below. Check results by graphic solution and by a check formula containing the unknown parts. Use the table of §8.

1. $\alpha = 35°$, $a = 100$.

2. $\alpha = 65°$, $b = 150$.

3. $\alpha = 48°$, $c = 75$.

4. $\beta = 33°$, $c = 50$.

5. $\beta = 58°$, $b = 750$.

6. $\beta = 15°$, $a = 0.15$.

7. $\alpha = 50°$, $c = 0.045$.

8. $\beta = 80°$, $c = 1.25$.

9. $\beta = 52°$, $a = 16\frac{2}{3}$.

10. $\alpha = 25°$, $b = 0.04$.

11. Find the length of chord subtended by a central angle of 110° in a circle of radius 50 ft. (First find the half-chord.)

12. Find the central angle subtended by a chord of 90 ft. in a circle of radius 200 ft.

13. Find the radius of the circle in which a chord of 120 ft. subtends an angle of 70°.

14. Find the length of side of a regular decagon inscribed in a circle of radius 300 ft.

15. Find the length of side of a regular pentagon circumscribed about a circle of radius 200 ft.

16. From a point in the same horizontal plane as the foot of a flag pole, and 200 ft. from it, the angle of elevation of the top is 20°. How high is the pole?

17. A vertical pole 35 ft. high casts a shadow 50 ft. long on level ground. Find the altitude of the sun.

18. If a road rises at an angle of 5°, how many feet does it rise in a distance of one mile measured along the road?

19. If the long arm of a carpenter's square is 24 inches, how far along the short arm should he place a mark so that the line from the mark to the far end of the long arm will make an angle of 22.5° with the long arm?

20. In Ex. 19 what would be the angle if the mark were placed on the short side $12\frac{1}{2}$ inches from the vertex of the right angle?

CHAPTER

II

VARIATION OF THE TRIGONOMETRIC FUNCTIONS

15. Variation of the sine function. Graph. Periodicity.

Suppose the point P of Fig. 5 to describe the circumference of the circle in such a way that angle XOP varies continuously from $0°$ to $360°$. Let us trace the changes in the ordinate MP or, what is the same thing, in the sine of angle XOP.

In quadrant I, MP or $\sin XOP$ increases from 0 to $+1$.
In quadrant II, MP or $\sin XOP$ decreases from $+1$ to 0.
In quadrant III, MP or $\sin XOP$ decreases from 0 to -1.
In quadrant IV, MP or $\sin XOP$ increases from -1 to 0.

To represent these changes graphically we shall take x to stand for the number of degrees in angle XOP and make a diagram showing the value of $\sin x$ for a selected set of values of angle x.

NOTE. It will be convenient here to use the letter x to represent our variable angle. This use of the letter should not be confused with its earlier use as the abscissa of a point.

In Fig. 13, below, the horizontal central line is the angle scale, on which one division is taken to represent $15°$ of angle, so that six divisions represent $90°$. On the angle scale, or x-axis, a distance measured to the right from O represents a positive angle x. The quadrantal values $x = 90°, 180°, 270°, 360°$ are represented by 6, 12, 18, 24 divisions respectively.

On the vertical scale we choose a convenient length to represent the sine of 90°, which is 1. This is subdivided into 5 divisions in the figure.

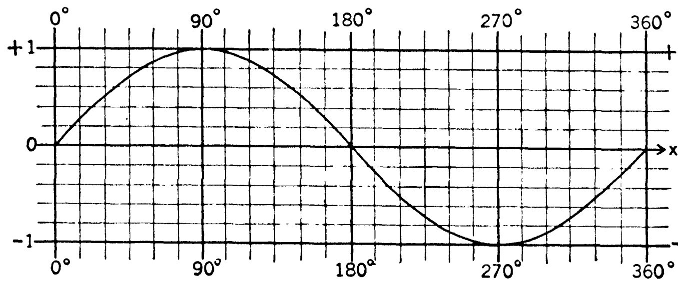


FIG. 13

At intervals of 15° on the angle scale, starting with $x = 0$, dots are placed above or below this scale, the height of each dot representing the value of $\sin x$. These values may be read off from Fig. 5. Joining the dots by a smooth curve gives us a graphic picture of the varying values of $\sin x$, as x changes from 0° to 360°. The approximate value of the sine of any angle can be read off at once from this *graph of* $\sin x$, commonly called the *sine curve.*

Periodicity. The sine curve has a simple wave form. By continuing it from 360° to 720° another wave would appear, and so on indefinitely. By taking negative values of x, to the left from 0°, these waves could be continued to the left.

A function of x, $f(x)$, which goes through the complete cycle of all its values when x ranges from $x = a$ to $x = a + h$, and again when x goes from $a + h$ to $a + 2h$, and so on, is called a *periodic function* with *period* h. In symbols,

$$f(x) = f(x + h) = f(x + 2h) = f(x \pm nh),$$

when n is any positive integer.

The function $\sin x$ has this character because

$$\sin x = \sin (x + 360°) = \sin (x \pm n \cdot 360°).$$

Therefore *$\sin x$ is a periodic function with period 360°.*

16. Variation of the cosine. Graph. Periodicity.

In Fig. 5 the abscissa OM gives the value of the cosine of angle XOP. In the notation of §15, $OM = \cos x$. We see that OM, or $\cos x$, varies from 1 to 0 in quadrant I, from 0 to -1 in quadrant II, from -1 to 0 in quadrant III, and from 0 to $+1$ in quadrant IV.

If we take the values of $\cos x$ for values of x at intervals of 15°, starting with $x = 0°$, and place dots to mark these values as was done for $\sin x$, we obtain the graph of $\cos x$, or the *cosine curve.*

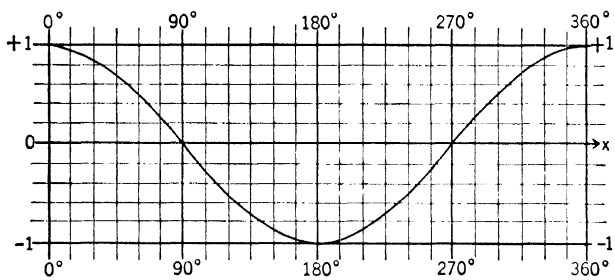


Fig. 14

This is a wave curve just like the sine curve, but with the crests of the wave 90° behind the crests of the sine wave. We say that the two waves differ in "*phase*" by 90°. See Fig. 15.

Periodicity. Just as for the sine function we have

$$\cos x = \cos (x + 360°) = \cos (x \pm n \cdot 360°).$$

Therefore, *cos x is a periodic function with period 360°.*

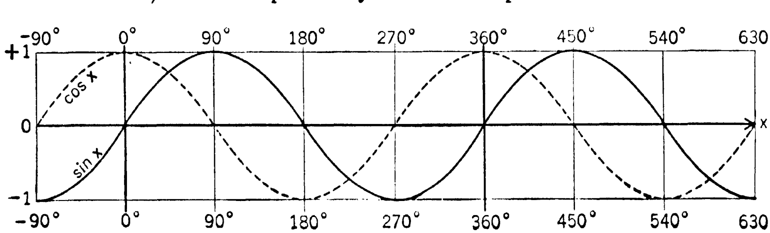


Fig. 15

For convenient comparison we show the graphs of both $\sin x$ and $\cos x$ on a single diagram.

EXERCISES

1. For what values of x is $\sin x = 0$? $\cos x = 0$?

2. For what values of x is $\sin x = \pm 1$? $\cos x = \pm 1$?

3. For what values of x is $\sin x = \cos x$?

17. Variation of sec x and csc x. Graphs. Periodicity.

From the definitions of $\sin x$ and $\csc x$ we have

$$\sin x = \frac{\text{ordinate}}{\text{distance}}, \qquad \csc x = \frac{\text{distance}}{\text{ordinate}}. \qquad \therefore \csc x = \frac{1}{\sin x}.$$

Hence when $\sin x = +1$ or -1, also $\csc x = +1$ or -1. As $\sin x$ decreases and approaches 0, $\csc x$ will increase and grow rapidly

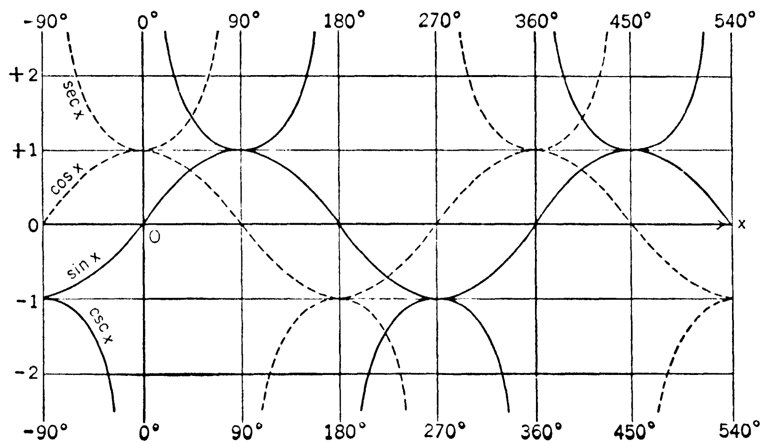


FIG. 16

larger in numerical value. Fig. 16 shows the graph of $\csc x$ and its relation to the graph of $\sin x$.

Likewise we have $\sec x = \dfrac{1}{\cos x}$, and Fig. 16 shows the graph of $\sec x$ in its relation to $\cos x$.

When $x = 0°$, $\sin x = 0$ and $\csc 0°$ has no value. When x is a small positive angle, as $x = 1°$, $\sin x$ is quite small and $\csc x$ is very large. As angle x approaches zero from the right, e.g. $x = 1°$, $x = 0.1°$, $x = 0.01°$, $x = 0.001°$, etc., $\csc x$ increases indefinitely. We say that $\csc x$ becomes positively infinite as x approaches 0 from the right and write $\csc\,(0° +) = +\infty$. When

x is a small negative angle, as $x = -1°$, csc x is represented by a large negative number. As x approaches 0° from the left csc x increases indefinitely in the negative direction; csc x becomes negatively infinite as x approaches 0° from the left and we write $\csc(0° -) = -\infty$. More briefly we write $\csc 0° = \pm\infty$ according as 0° is approached from the right or the left. A similar situation exists at 180° and at all other *even* multiples of 90°, positive or negative.

In the same way we are led to write $\sec(90° -) = +\infty$ and $\sec(90° +) = -\infty$, or, more briefly, $\sec 90° = \pm\infty$; similarly at all *odd* multiples of 90°.

NOTE. It should be carefully noted that the symbol ∞ is not a number, and that the statement $\csc 0° = \pm\infty$ does not assign a value to csc 0°. It merely indicates that, as angle x approaches 0°, csc x increases or decreases without limit.

18. Variation of tan x and cot x. Graphs. Periodicity.

In quadrant I, tan x starts at 0 when $x = 0$, becomes 1 at 45° and increases rapidly and without bound as x approaches 90°. Just after $x = 90°$ tan x has a large negative value, becomes -1 at $x = 135°$ and 0 at $x = 180°$. In quadrant III the values in

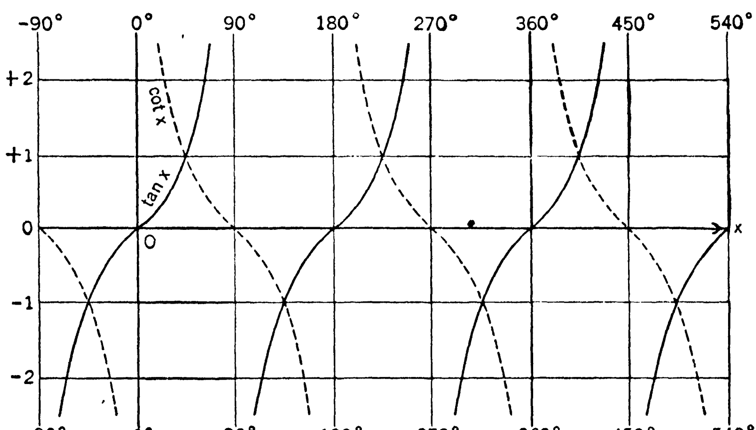


FIG. 17

quadrant I are repeated; in quadrant IV, the values in quadrant II are repeated. Similarly we can trace the changes in cot x.

The graphs of these two functions are shown in Fig. 17. Because $\cot x = 1 \div \tan x$, either function has a large value when the other has a small value. The two functions are both positive or both negative, according to the quadrant of the angle.

Periodicity. The functions $\tan x$ and $\cot x$ are periodic, with period 180°.

We have $\tan x = \tan (x + 180°) = \tan (x \pm n \cdot 180°)$.

Similarly for $\cot x$.

As x approaches 90° $\tan x$ increases (or decreases) without limit. We write $\tan 90° = \pm \infty$. Also $\tan 270° = \pm \infty$, etc. Likewise $\cot 0° = \pm \infty$, $\cot 180° = \pm \infty$, etc.

Exercise. Make a chart showing all six of the trigonometric functions on one diagram. Dotted lines, or lines of different colors, may be used to distinguish the different curves.

19. Relations between the functions of an angle.

From the general definitions of the functions given in §4, putting angle $XOP = x$, we find that

$$\sin x = \frac{1}{\csc x}; \quad \cos x = \frac{1}{\sec x}; \quad \tan x = \frac{1}{\cot x}.$$

$$\tan x = \frac{\text{ordinate}}{\text{abscissa}} = \frac{\dfrac{\text{ordinate}}{\text{distance}}}{\dfrac{\text{abscissa}}{\text{distance}}} = \frac{\sin x}{\cos x}; \quad \cot x = \frac{\cos x}{\sin x}.$$

Also, whatever be the quadrant of angle $XOP = x$ (figure of §4), we have

$$(\text{ordinate})^2 + (\text{abscissa})^2 = (\text{distance})^2.$$

Dividing this equation through in turn by $(\text{distance})^2$, $(\text{abscissa})^2$, and $(\text{ordinate})^2$, and expressing the resulting ratios as functions we have

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1, \\ 1 + \tan^2 x &= \sec^2 x, \\ 1 + \cot^2 x &= \csc^2 x. \end{aligned}$$

All the above relations between the functions of angle x are true for all values of x. They form a first set of working formulas, and should be thoroughly committed to memory. They are collected below, as

Formulas, Group A

(1) $\sin x = \dfrac{1}{\csc x}$.

(2) $\cos x = \dfrac{1}{\sec x}$.

(3) $\tan x = \dfrac{1}{\cot x}$.

(4) $\tan x = \dfrac{\sin x}{\cos x}$.

(5) $\cot x = \dfrac{\cos x}{\sin x}$.

(6) $\sin^2 x + \cos^2 x = 1$.

(7) $1 + \tan^2 x = \sec^2 x$.

(8) $1 + \cot^2 x = \csc^2 x$.

We shall apply these formulas in two examples.

Example 1.

Prove that $\tan x + \cot x = \sec x \csc x$.

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$
$$= \frac{1}{\sin x \cos x} = \frac{1}{\sin x} \cdot \frac{1}{\cos x} = \csc x \sec x.$$

Example 2.

Prove that

$$\frac{\csc x}{\tan x + \cot x} = \cos x.$$

$$\frac{\csc x}{\tan x + \cot x} = \frac{\csc x}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}$$
$$= \frac{\csc x}{\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}}$$
$$= \frac{\csc x}{\dfrac{1}{\sin x \cos x}} = \csc x \sin x \cos x = \cos x.$$

In both examples all the steps taken are true for all values of x, since this is true of all the formulas of group A. Hence the given equations are true for all values of x for which the functions are defined, and they are therefore called *trigonometric identities.*

The equation $\sin^2 x - \cos^2 x = 1$ is not true for all values of x, but holds only for certain special values; it is not an identity.

20. EXERCISES 5

Prove the following identities:

1. $\sin x \cot x = \cos x.$
2. $\dfrac{1}{\tan x \csc x} = \cos x.$
3. $\dfrac{\sec x}{\tan x} = \csc x.$
4. $\dfrac{\csc x}{\cot x} = \sec x.$
5. $(\sin^2 x + \cos^2 x)^2 = 1.$
6. $\dfrac{\sin \theta}{\cos \theta \cot \theta} = \tan^2 \theta.$

(For names of Greek letters see first page of appendix.)

7. $(\csc \theta - \cot \theta)(\csc \theta + \cot \theta) = 1.$
8. $(\sec x - \tan x)(\sec x + \tan x) = 1.$
9. $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta.$
10. $\sin^2 \alpha + \cos^2 \alpha = \csc^2 \alpha - \cot^2 \alpha.$
11. $(\sin \alpha - \cos \alpha)^2 = 1 - 2 \sin \alpha \cos \alpha.$
12. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x.$
13. $(1 - \cos^2 x) \sec^2 x = \tan^2 x.$
14. $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta.$
15. $\sec \theta \csc \theta - \cot \theta = \tan \theta.$
16. $\cot \varphi \cos \varphi + \sin \varphi = \csc \varphi.$
17. $\cos^2 \varphi \csc^2 \varphi = \csc^2 \varphi - 1.$
18. $\dfrac{\sin \varphi}{1 + \cos \varphi} = \dfrac{1 - \cos \varphi}{\sin \varphi}.$
19. $\dfrac{1 + \tan^2 \beta}{1 + \cot^2 \beta} = \dfrac{\sin^2 \beta}{\cos^2 \beta}.$
20. $(1 - \cos^2 \beta)(1 + \cot^2 \beta) = 1.$
21. $\tan^4 x - \sec^4 x = 1 - 2 \sec^2 x.$
22. $\dfrac{\cos x + \sin x}{\cos x - \sin x} = \dfrac{1 + \tan x}{1 - \tan x}.$
23. $(\tan x - 1)(\cot x - 1) = 2 - \sec x \csc x.$
24. $\csc \theta + \cot \theta = \dfrac{\sin \theta}{1 - \cos \theta}.$
25. $(a \cos x - b \sin x)^2 + (a \sin x + b \cos x)^2 = a^2 + b^2.$
26. $\cos^2 \varphi + (\sin \varphi \cos \theta)^2 + (\sin \varphi \sin \theta)^2 = 1.$
27. $\tan \alpha + \tan \beta = \tan \alpha \tan \beta (\cot \alpha + \cot \beta).$

21. The functions of any angle in terms of the functions of an acute angle.

It is possible to express in a simple manner any function of any angle in terms of the proper function of an acute angle. Then a table of the values of the functions of angles from 0° to 90° will serve for all angles. In fact, in view of §12, a table of functions of angles from 0° to 45° would be sufficient, though not convenient.

I. Any angle, positive or negative, can be made to correspond to a positive acute angle by adding to it, or subtracting from it, an integral multiple of 90°.

Examples.

(a) $780° - 8 \times 90° = 60°$. (c) $510° - 5 \times 90° = 60°$.
(b) $-480° + 6 \times 90° = 60°$. (d) $-750° + 9 \times 90° = 60°$.

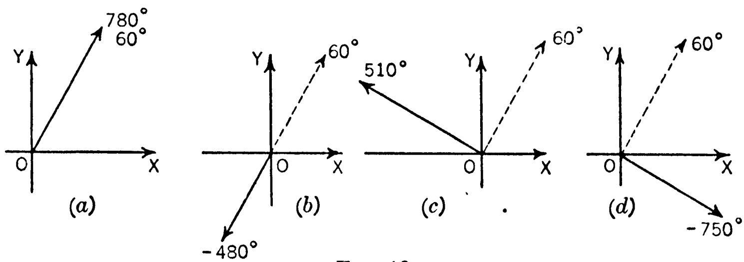


FIG. 18

In what follows, we designate the original angle by θ (theta) and the new angle by θ'.

II. Let OP be the terminal line of a given angle θ. (Fig. 19) When angle θ is changed by an *even* multiple of 90° the terminal line of the new angle, θ', will coincide with OP or with

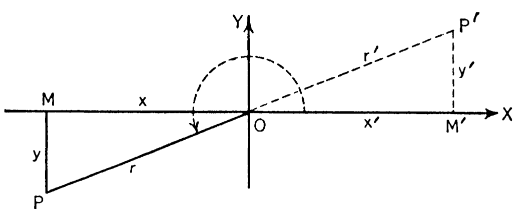


FIG. 19

its continuation OP'. In the first case the angles θ and θ' have the same terminal line and hence the same set of function values. In the second case the functions of θ' are determined by $\triangle\ OM'P'$ which is *directly similar* to $\triangle\ OMP$, ordinate corresponding to ordinate and abscissa to abscissa. Therefore any trigonometric ratio from $\triangle\ OMP$ will have the same numerical value as the corresponding ratio from $\triangle\ OM'P'$, but may differ from it in sign.

RULE (a). *Any function of an angle θ is* ***numerically*** *equal to the* ***same*** *function of θ', when θ' differs from θ by an* ***even*** *multiple of $90°$.*

In symbols, if f stands for any one of the six functions,

$$f(\theta) = \pm f(\theta') \text{ where } \theta' = \theta \pm n \times 90°; \ n \text{ even.}$$

When the new angle θ' is an acute angle (first quadrant) choose the sign before $f(\theta')$ $+$ or $-$ according as the function of the original angle θ is $+$ or $-$.

Examples.

(The student should draw illustrative figures.)

1. $600° - 6 \times 90° = 60°$; sin $600°$ is negative and tan $600°$ is positive.
$\therefore \sin 600° = -\sin 60°$; $\tan 600° = +\tan 60°$.

2. $-510° + 6 \times 90° = 30°$; sec $(-510°)$ is $-$ and cot $(-510°)$ is $+$.
$\therefore \sec(-510°) = -\sec 30°$; $\cot(-510°) = \cot 30°$.

III. Again let OP be the terminal line of a given angle θ. In Fig. 20 angle θ is taken to be in quadrant II.

When angle θ is changed by an *odd* multiple of $90°$ the terminal

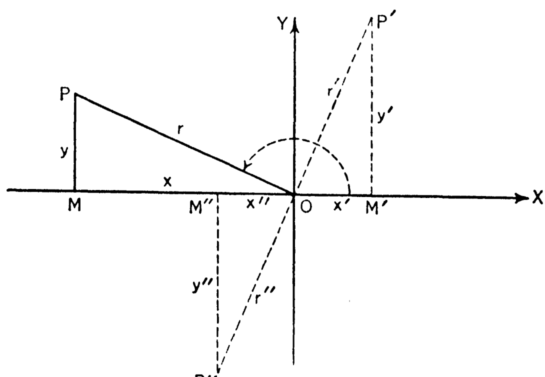


FIG. 20

line of the new angle, θ', will lie at right angles to OP, in the direction OP' or OP''. If we take θ' as the first quadrant angle XOP', we note that $\triangle\, OM'P'$ is *inversely similar* to $\triangle\, OMP$, in the sense that abscissa x' corresponds to ordinate y, and ordinate y' is homologous to abscissa x. Hence any function of

θ is *numerically* equal to the *co-function* of θ'. Exactly the same is true if we take θ' as having the terminal line OP''. So we have

RULE (b). *Any function of an angle θ is* ***numerically*** *equal to the* ***cofunction*** *of θ', when θ' differs from θ by an* ***odd*** *multiple of $90°$.*

In symbols, if f stands for any one of the six functions,

$$f(\theta) = \pm \text{co-}f(\theta'), \text{ where } \theta' = \theta \pm n \times 90°; n \text{ odd}.$$

When θ' is an acute angle (first quadrant) choose the sign before co-$f(\theta')$ $+$ or $-$ according as the function of the original angle θ is $+$ or $-$.

Examples.

(The student should check by drawing figures.)

1. $680° - (7 \times 90°) = 50°$. $\sin 680° = -\cos 50°$; $\tan 680° = -\cot 50°$.

2. $-390° + (5 \times 90°) = 60°$. $\cos(-390°) = \sin 60°$; $\cot(-390°) = -\tan 60°$.

22. EXERCISES 6

Express all the functions of the following angles in terms of functions of acute angles:

1. 140°.	**5.** 355°.	**9.** − 318°.	**13.** − 1040°.
2. 155°.	**6.** − 35°.	**10.** 738°.	**14.** − 410°.
3. 235°.	**7.** − 115°.	**11.** − 670°.	**15.** 535°.
4. 335°.	**8.** − 255°.	**12.** 1120°.	**16.** − 103°.

Express all the functions of the following angles in terms of functions of angles between 0° and 45°.

17. 75°.	**19.** 110°.	**21.** − 335°.	**23.** 790°.
18. − 80°.	**20.** 255°.	**22.** 600°.	**24.** − 510°.

Give the exact values of the functions of:

25. 120°.	**29.** − 30°.	**33.** − 240°.
26. 135°.	**30.** − 45°.	**34.** 315°.
27. 150°.	**31.** − 60°.	**35.** 600°.
28. 300°.	**32.** − 120°.	**36.** − 510°.

23. Relations between the functions of $+\theta$ and $-\theta$.

The figure is drawn for angle θ in the first quadrant. Taking equal distances on the terminal lines of $+\theta$ and $-\theta$ and drawing the ordinates, we have two triangles with a common abscissa and ordinates numerically equal but of opposite signs.

Comparing the trigonometric ratios of $-\theta$ with those of $+\theta$ we see that

$\sin(-\theta) = -\sin\theta;$
$\csc(-\theta) = -\csc\theta;$
$\tan(-\theta) = -\tan\theta;$
$\cot(-\theta) = -\cot\theta;$
$\cos(-\theta) = \cos\theta;$
$\sec(-\theta) = \sec\theta.$

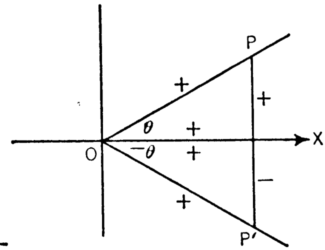


FIG. 21

RULE. *The cosine and secant remain unchanged when the sign of the angle is changed; the other four functions change sign when the sign of the angle is changed.*

Exercise. Draw a figure and show that these equations are true when θ is in the second quadrant; in the third quadrant; in the fourth quadrant.

24. EXERCISES 7

For the following angles draw figures to verify the rule of §23. Where possible give the exact values of the functions.

1. − 45°.	**5.** − 120°.	**9.** − 103°.	**13.** − 225°.
2. − 30°.	**6.** − 150°.	**10.** − 35°.	**14.** − 410°.
3. − 60°.	**7.** − 135°.	**11.** − 255°.	**15.** − 1040°.
4. − 90°.	**8.** − 115°.	**12.** − 75°.	**16.** − 318°.

25. Versed sine, coversed sine, haversine.

The three expressions $1 - \cos\theta$, $1 - \sin\theta$, $\frac{1}{2}(1 - \cos\theta)$ occur often enough in the applications of trigonometry to warrant the use of special symbols for them. These are

$1 - \cos\theta$ = versed sine of θ = vers θ;
$1 - \sin\theta$ = coversed sine of θ = covers θ;
$\frac{1}{2}(1 - \cos\theta)$ = haversine of θ = hav θ.

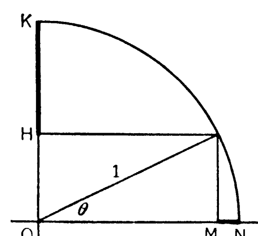


FIG. 22

In the figure, θ being an acute angle, vers $\theta = MN$ because $MN = ON - OM$ and $ON = 1$, $OM = \cos\theta$. So vers θ represents the "rise" of an arc above its chord in a unit circle.

EXERCISES 8

Find the values of vers θ, covers θ and hav θ for the following angles.

1. 30°.	**4.** 90°.	**7.** 150°.	**10.** − 225°.
2. 45°.	**5.** 120°.	**8.** − 30°.	**11.** − 300°.
3. 60°	**6.** 135°.	**9.** − 120°.	**12.** − 315°.

CHAPTER

III

RADIAN MEASURE. APPLICATIONS. USE OF TABLES OF NATURAL FUNCTIONS.

26. Radian measure.

The **degree** is an artificial unit for the measurement of angles. In France, where at the time of the Revolution an attempt was made to put all measurements on the basis of the decimal scale, the quadrant of the circle was divided into 100 equal parts and the angle subtended at the center by one part was called a **grade.** Each grade was then subdivided into 100 equal parts called **minutes,** and each minute into 100 **seconds.** The degree and the grade are thus two arbitrary units for the measurement of angles, and any number of such units might be chosen.

In the artillery service a common unit of angle is the **mil,** so chosen that 1600 mils make a quadrant of 90°. This will be discussed in Chapter VII.

There is one unit which is naturally related to the circle, and which is as commonly used in theory as the degree in practice. *It is the central angle subtended by an arc equal in length to the radius of the circle, and is called a* ***radian*** (figure).

Since the circumference contains the radius 2π times, the entire central angle of 360° contains 2π radians, i.e.,

$$2\pi \text{ radians} = 360°.$$

Hence,

π radians $= 180°$;

$\frac{\pi}{2}$ radians $= 90°$;

$\frac{\pi}{4}$ radians $= 45°$; and so on.

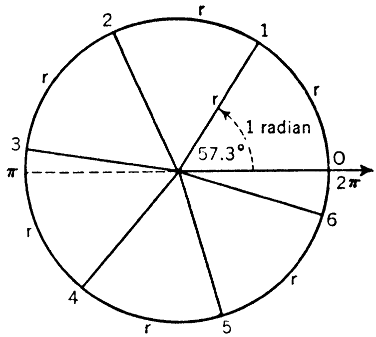


FIG. 23

In dealing with angles measured in radians it is customary to omit specifying the unit used; it is understood that when no unit is indicated the radian is implied. Thus,

$$2\pi = 360°, \ \pi = 180°, \ \frac{\pi}{3} = 60°, \ 2\tfrac{1}{2} = 2\tfrac{1}{2} \text{ radians, and so on.}$$

NOTE. To get the standard form of the graphs of the equations $y = \sin x$, $y = \cos x$, etc., take x in radians on the x-axis, thus: $x = 0.1, 0.2, 0.3, \ldots, 1, \ldots$ and find the corresponding values of y; use the same unit of length for both x and y.

27. Radians into degrees, and conversely.

Since 2π (radians) $= 360°$,

$$\text{therefore, } 1 \text{ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = \frac{180°}{3.1416-} = 57.3+°;$$

$$\text{also, } \quad 1 \text{ degree} = \frac{2\pi}{360}(\text{radians}) = \frac{\pi}{180}(\text{radians})$$

$$= \frac{1}{57.3+}(\text{radians}) = 0.017+ \ (\text{radians}).$$

RULE: *To convert radians into degrees, multiply the number of radians by* $\frac{180}{\pi}$ *or 57.3+.*

To convert degrees into radians, multiply the number of degrees by $\frac{\pi}{180}$ *or* $\frac{1}{57.3+}$ *or 0.017+.*

By taking a sufficiently accurate value of π, we find,

1 radian $= 57.2957795° = 3437.74677' = 206264.8''$.
$1° = 0.0174533$ radians.
$1' = 0.0002909$ radians (point, 3 ciphers, 3, approx.).
$1'' = 0.0000048$ radians (point, 5 ciphers, 5, approx.).

The measure of an angle in radians is often called the **circular measure** of the angle.

Examples.

1. Express 240° in radians.

$$240° = 240 \times \frac{\pi}{180} \text{ radians} = \frac{4\pi}{3} \text{ radians}.$$

2. Express in degrees the angle whose radian measure is $1 + \pi$.

$$(1+\pi) \text{ radians} = (1+\pi) \times \frac{180}{\pi} \text{ degrees} = \left(\frac{180}{\pi} + 180\right) \text{ degrees}$$
$$= 57.3° + 180° \qquad = 237.3° +.$$

3. Express in degrees the angle whose circular measure is $\frac{2}{\pi - 1}$ radians.

We can see that, since $\pi = 3.14+$, the given fraction has a value a little less than 1; hence the angle is a little less than one radian, hence less than 57.3°. Making the reduction we have

$$\frac{2}{\pi - 1} \text{ radians} = \frac{2}{\pi - 1} \times \frac{180}{\pi} \text{ degrees} = \frac{360}{\pi^2 - \pi} \text{ degrees}$$
$$= \frac{360}{6.73\pm} \text{ degrees} = 53.5°\pm.$$

See also Table V, Appendix.

28. EXERCISES 9

Express in degrees, minutes and seconds the angles whose radian measures are:

1. $\frac{\pi}{12}, \frac{5\pi}{3}, \frac{5\pi}{16}, \frac{8\pi}{5}, \frac{22\pi}{15}$.

2. $2, 1.5, \frac{3}{2}, \frac{4}{3}, \frac{8}{5}$.

3. $\frac{5\pi}{12}, -\frac{3}{2}, \frac{2\pi}{15} + 1, \frac{\pi}{3} - \frac{2}{3}, \frac{2\pi + 5}{8}$.

4. $\frac{1}{4} + \pi, \frac{\pi}{4} - \frac{1}{3}, \frac{1}{\pi}, \frac{2}{\pi - 3}, \pi^2$.

5. $\frac{\pi}{\pi^2 + 1}, \frac{\pi^2}{1 - \pi}, \frac{\pi + 1}{\pi - 1}$.

Reduce the following angles to circular measure:

6. 30°, 120°, 150°, 225°, – 60°.
7. 375°, – 22½°, 187.5°, 106°, 93° 45′.
8. 85°, 191° 15′, 5° 37′ 30″, 90° 37′ 30″.
9. 10′, 10″. 0.1″, 12° 5′ 4″, 21° 36′ 8.1″.

29. Circular arc, sector, segment.

Let θ be the radian measure of the central angle subtended by an arc of length a in a circle of radius r.

(a) Now, in a given circle, arcs are proportional to their central angles; also, the whole circumference subtends at the center an angle of 360° or 2π radians. Therefore (Fig. 24)

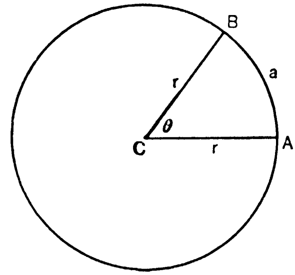


FIG. 24

$$\frac{\text{arc } AB}{\text{circumference}} = \frac{a}{2\pi r} = \frac{\theta}{2\pi}.$$

Therefore $\quad \boldsymbol{a = r\theta}.$

The length of a circular arc equals the product of the radius times the central angle (in radians).

(b) Also, in a given circle, the areas of sectors are proportional to the central angles of the sectors. Therefore, if S = area of sector ACB,

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{S}{\pi r^2} = \frac{\theta}{2\pi}. \qquad \text{Therefore } \boldsymbol{S = \tfrac{1}{2}r^2\theta}.$$

The area of a circular sector equals the product of one half the square of the radius times the central angle (in radians).

(c) In Fig. 24a we have a **segment** $ADBA$ of a circle cut off by chord AB. Then

$$\text{area of segment} = \text{area of sector} - \text{area of triangle}.$$
$$\text{area of sector} = \tfrac{1}{2}r^2\theta. \qquad (\theta \text{ in radians.})$$

To find the area of $\triangle CAB$, let BE be drawn perpendicular to CA. Then $BE = r \sin \theta$.

$$\begin{aligned}\text{area of } \triangle CAB &= \tfrac{1}{2} \text{ base times altitude} \\ &= \tfrac{1}{2}\, CA \cdot EB \\ &= \tfrac{1}{2}\, r \cdot r \sin\theta = \tfrac{1}{2}\, r^2 \sin\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{area of segment} &= \tfrac{1}{2}\, r^2\theta - \tfrac{1}{2}\, r^2 \sin\theta \\ &= \tfrac{1}{2}\, r^2 (\theta - \sin\theta).\end{aligned}$$

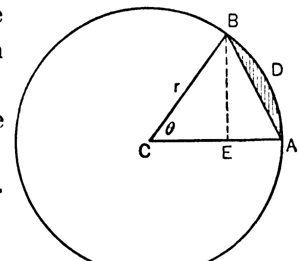


FIG. 24a

EXERCISES 10

1. In a circle of radius 12 inches a chord is drawn 6 inches from the center. Calculate the length of the chord and of the arc and the area of the segment.

2. Determine the area between a circumference of radius 10 inches and a regular inscribed pentagon.

3. The area of a sector is 50 square inches and its central angle is 2 radians. Find the radius of the circle.

30. Angular and linear displacement; angular and linear speed.

When we say that a wheel rotates at the rate of 10 R.P.M. (revolutions per minute) we mean that a given radius of the wheel would turn through an angle of $10 \times 360°$ or $3600°$ in one minute if the rate of rotation remains constant during that minute. For rate of rotation we commonly use the term *angular speed*, and designate it by the Greek letter omega, ω.

When the rate of rotation, or angular speed, is 10 R.P.M. we write

$$\omega = 10 \text{ R.P.M., or } \omega = 10 \text{ rev. per min., or } \omega = 10 \text{ rev./min.}$$

This is equivalent to any of the following:

$$\omega = 60 \times 10 \text{ rev. per hour} = 600 \text{ rev./hr.}$$
$$\omega = \tfrac{1}{60} \times 10 \text{ rev. per sec.} = \tfrac{1}{6} \text{ rev./sec.}$$
$$\omega = 10 \times 360° \text{ per min.} = 3600 \text{ degr./min.}$$
$$\omega = 10 \times 2\pi \text{ rad. per min.} = 20\pi \text{ rad./min.}$$
$$\omega = 60 \text{ degr./sec.} \qquad \omega = \frac{\pi}{3} \text{ rad./sec.}$$

Then the same angular speed may be indicated by many different numbers, depending on the unit of angle and the unit of time.

The angle θ through which a given radius of the wheel turns in t units of time will be $t \times \omega$. This is called the angular displacement in time t; $\theta = \omega t$.

Angular displacement = angular speed × time.

Examples.

If $\omega = 30$ degr./sec. and $t = 10$ sec., $\theta = 300°$.

If $\omega = \frac{\pi}{2}$ rad./min. and $t = 20$ min., $\theta = 10\pi$ radians.

Suppose that we follow the motion of a point on the rim of a wheel rotating with constant speed. Let P be the point, r its distance from the center, ω the angular speed in radians per unit of time, and θ the angle in radians turned through in t units of time.

If the wheel rotates through angle θ in time t we have

$$\theta = \omega t \quad \text{and} \quad r\theta = r\omega t.$$

So we see that

$r\theta$ = *arc* AP = *linear displacement* of P in time t.

$r\omega$ = displacement of P in a *unit* of time.

= *linear speed* of P.

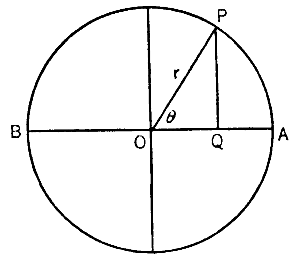


FIG. 25

Example 1.

A wheel 4 feet in diameter is rotating with uniform angular speed of π radians per second. What is the linear speed of a point on the rim? How far will such a point travel in 10 seconds?

Here $r = 2$ feet, $\omega = \pi$ rad./sec., $t = 10$ sec.
Linear speed of $P = r\omega = 2\pi$ ft./sec.
Linear displacement of $P = r\theta = r\omega t = 20\pi$ feet.

Example 2.

Suppose point P, Fig. 25, to be moving with uniform speed of 20 feet per second in a circle of radius 5 feet. What is its angular speed, and what is its angular displacement in time t seconds?

In t seconds P moves through an arc of $20t$ feet. Central angle $\theta = \text{arc} \div \text{radius} = 4t$ radians.
But $\theta = \omega t$. Therefore $\omega = 4$ radians per second.

Simple harmonic motion. As P moves uniformly around the circle point Q, which is the foot of the perpendicular from P on BA (Fig. 25), moves back and forth along the diameter AB. Its distance from O is $OQ = r \cos \theta = r \cos \omega t$.

Point Q will move slowly when θ is near 0°, it will increase its speed as θ becomes 90°, and then diminish its speed as θ nears 180°. This cycle will be reversed as θ varies from 180° to 360°.

DEFINITION. Point Q is said to have *simple harmonic motion.*

31. EXERCISES 11

1. If θ is the degree measure of a central angle, show that

$$a = \text{arc} = \frac{\pi}{180} \times r \times \theta \quad \text{and} \quad S = \text{sector} = \frac{\pi}{360} \times r^2 \times \theta.$$

2. If $r = 100$ inches, find the length of arc and area of sector (a) when $\theta = 1$ (radian). (b) When $\theta = 0.5$. (c) When $\theta = 1.5$. (d) When $\theta = 30°$. (e) When $\theta = 75°$.

3. Find the central angle (a) when $r = 100$ and $a = 25$. (b) When $r = 100$ and $a = 125$. (c) When $r = 100$ and $S = 1000$. (d) When r is 100 and $S = 100$. In each case give the value of θ in radians and also in degrees.

4. Taking the radius of the earth as 3960 miles calculate the number of feet in an arc of a meridian whose central angle is $1'$. This is the *nautical mile.*

Show that the nautical mile is about one seventh longer than the statute or land mile.

5. In a circle of radius 100 inches a chord is drawn at a distance of 80 inches from the center. Find the length of the chord and of its subtended arc. Find the area of the segment formed by this chord and its arc.

6. A cylindrical gasoline tank 12 feet long and 4 feet in diameter lies on its side in a horizontal position. Measurement shows that the depth of the gasoline at the center is 16 inches. How many gallons of gasoline are there in the tank?

7. To a circle of radius 100 inches tangents are drawn at two points separated by an arc 50 inches long. Find the angle between these tangents.

In Fig. 24a the following quantities appear:
AC, AB, angle ACB, arc ADB, sector $CADBC$, triangle ABC, segment $ADBA$.

8. Calculate each of the other quantities when $AC = 50$ and $AB = 40$.

9. As in Ex. 8 when $AC = 50$ and arc $AB = 20$.

10. How many radians are there in the central angle subtended by one side of a regular inscribed decagon?

11. How many radians in the central angle subtended by an arc of 150 feet in a circle of radius 50 feet?

12. A wheel makes 1000 revolutions a minute. Find its angular speed in radians per second.

13. How many revolutions per minute are equivalent to an angular speed of 3π rad./sec.?

14. What is the angular speed if a point on the rim of a wheel of radius 10 inches moves with a linear speed of 25 inches per second? Give the answer in radians per second and also in revolutions per minute.

15. If a turbine wheel is 8 feet in diameter how fast would it have to rotate to cause a point on the rim to move with the speed of sound in air (1080 ft./sec.)?

16. In Example 2 of §30 calculate the length of OQ at intervals of 0.1 sec., from $t = 0$ to $t = 1$.

32. Use of tables of natural trigonometric functions.

Such tables give the values of the functions of angles from 0° to 90°. But they will serve for all angles since any function of any angle is reducible to a function of an acute angle.

Table III of the Appendix gives the values, to 4 decimal places, of the six functions of angles from 0° to 90°, at intervals of 10′. For intermediate angles we obtain the function values by *interpolation.*

Such tables are used in two ways.

(a) *Directly.* Given the angle to find the numerical value of one of its functions.

(b) *Inversely.* Given the numerical value of a function to find the corresponding angles.

We shall illustrate the direct use of the tables by examples. The tables give only four decimal places; therefore answers are given only to four decimal places. Note that angles read down on the left from 0° to 45° and up on the right from 45° to 90°. The names of functions at the top of the page apply to the angles at the left, those at the bottom of the page to the angles at the right. Our examples will include angles from the various quadrants, including negative angles.

We use the principle of *linear interpolation,* that is, we assume that, for *sufficiently small* changes in the angle, *the change in the function is proportional to the change in the angle.* This principle does not apply to some of the functions of angles near 0° or near 90°.

1. $\sin 21^\circ 13' = ?$

$$\left.\begin{aligned}\sin 21^\circ 10' &= 0.3611\\ \sin 21^\circ 20' &= 0.3638\end{aligned}\right\} \quad \text{diff.} = 0.0027$$

$$\begin{aligned}\sin 21^\circ 13' &= 0.3611 + 0.3\ (0.0027)\\ &= 0.3619\end{aligned}$$

2. $\cos 70^\circ 32' = ?$

$$\left.\begin{aligned}\cos 70^\circ 30' &= 0.3338\\ \cos 70^\circ 40' &= 0.3311\end{aligned}\right\} \quad \text{diff.} = -\ 0.0027$$

$$\begin{aligned}\cos 70^\circ 32' &= 0.3338 - 0.2(0.0027)\\ &= 0.3333\end{aligned}$$

3. $\tan 150° 15' = ?$

$150° 15' - 1 \times 90° = 60° 15'.$

$\tan 150° 15' = -\cot 60° 15'.$ (§21)

$$\left.\begin{array}{l}\cot 60° 10' = 0.5735\\ \cot 60° 20' = 0.5696\end{array}\right\} \text{diff.} = -0.0039$$

$\cot 60° 15' = 0.5735 - .5(0.0039) = 0.5715.$

$\tan 150° 15' = -0.5715.$

4. $\tan(-150° 15') = ?$

$-150° 15' + 2 \times 90° = 29° 45'.$

$\tan(-150° 15') = \tan 29° 45' = 0.5715.$ (§21)

5. $\csc(-400° 43') = ?$

$-400° 43' + 5 \times 90° = 49° 17'.$

$\csc(-400° 43') = -\sec 49° 17'.$ (§21)

$$\left.\begin{array}{l}\sec 49° 10' = 1.5294\\ \sec 49° 20' = 1.5346\end{array}\right\} \text{diff.} = 0.0052$$

Now we may add .7 of the difference to 1.5294 or subtract .3 of the difference from 1.5346. The latter way is preferable. We get

$\sec 49° 17' = 1.5346 - 0.0016 = 1.5330.$

$\csc(-400° 43') = -1.5330.$

Example 6.

$\sec\left(\frac{4\pi}{7}\right) = ?$

$\frac{4\pi}{7}$ radians $= \frac{4}{7}\,180° = 102° 51.4'.$

$\sec 102° 51.4' = \sec(90° + 12° 51.4') = -\csc 12° 51.4' = -4.5042.$

Example 7.

$\cos\left(3 - \frac{4\pi}{7}\right) = ?$

$\left(3 - \frac{4\pi}{7}\right)$ radians $= 3$ radians $- \frac{4\pi}{7}$ radians

$= 171° 53.2' - 102° 51.4' = 69° 1.8'.$

$\cos 69° 1.8' = 0.3579.$

Note. In these examples we have systematically followed Rules (a) and (b) of §21. Other procedures may be followed. The angle $-150° 15'$ can be brought into the first quadrant by changing its sign (§23) and then reducing by 90° (§21). That is, we go from $-150° 15'$ to $+150° 15'$, then to $60° 15'$. To determine $\tan(-150° 15')$ we would have

$$\tan(-150° 15') = -\tan(150° 15') = \cot 60° 15'.$$

33. EXERCISES 12

Determine the values of sine, tangent and secant of each of the following angles. For Exercises 1–12 use a four-place table, for the rest use a five-place table.

1. $32° 25'$.	**8.** $-98° 18'$.	**15.** $61° 53' 15''$.	**21.** $\frac{3\pi}{7}$.
2. $17° 42'$.	**9.** $-122° 25'$.	**16.** $-8° 18' 40''$.	**22.** $1 + \frac{\pi}{5}$.
3. $61° 53'$.	**10.** $287° 42'$.	**17.** $\frac{\pi}{7}$.	**23.** $\frac{2\pi}{11}$.
4. $8° 18'$.	**11.** $511° 53'$.	**18.** $\pi - 1$.	**24.** $1 - \frac{\pi}{7}$.
5. $122° 25'$.	**12.** $548° 18'$.	**19.** $2 + \pi$.	
6. $-17° 42'$.	**13.** $122° 25.7'$.	**20.** $3\pi - 2$.	
7. $241° 53'$.	**14.** $17° 42.3'$.		

34. All angles corresponding to a given value of a function.

We have here the problem of the *inverse* use of a table like Table III, referred to in §33.

When a given value is assigned to one of the functions, as $\sin \theta = \frac{1}{2}$, there will in general be *two* possible positions of the terminal line, and only two. Exceptions occur when the terminal position falls on one of the quadrant lines, when there may be only *one* possible position. An angle whose terminal line falls on a quadrant line we call a *quadrantal angle*.

These statements are illustrated in the figures below. In each case we denote by θ_1 and θ_2 the two ***basic angles,*** that is, those angles obtained by the least possible rotation from the initial line OX.

Given: $\sin \theta = \frac{1}{2}$. $\sin \theta = -\frac{1}{2}$. $\tan \theta = 1$.

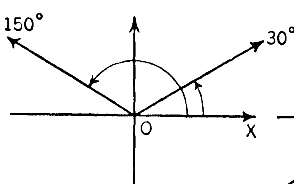


FIG. 26a

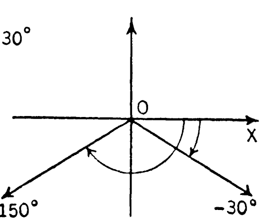


FIG. 26b

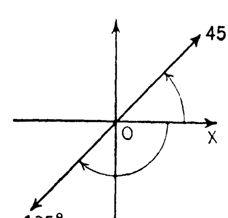


FIG. 26c

Quadrants:	I or II.	III or IV.	I or III.
Basic angles:	$\theta_1 = 30°$, $\theta_2 = 150°$.	$\theta_1 = -30°$, $\theta_2 = -150°$.	$\theta_1 = 45°$, $\theta_2 = -135°$.

An illustration of an exceptional case is furnished by $\sin\theta = 1$. Here there is only one possible position of the terminal line, with basic angle $\theta_1 = 90°$.

To determine all the angles for which $\sin\theta = \frac{1}{2}$ we need merely to write down expressions representing all angles coterminal with the basic angles 30° and 150°. These angles can differ from 30° or 150° only by an integral number of complete revolutions.

If n is an integer, positive or negative, any number of complete revolutions can be expressed by $n \cdot 360°$ or by $n \cdot 2\pi$ radians.

Therefore all solutions of the equation $\sin\theta = \frac{1}{2}$ are given by

$$\theta = 30° + n \cdot 360° \quad \text{or} \quad 150° + n \cdot 360°;$$

or by

$$\theta = \frac{\pi}{6} + 2n\pi \qquad \text{or} \quad \frac{5\pi}{6} + 2n\pi.$$

Here n may be any integer, positive or negative, or 0; $n = 0$ gives the basic angles, 30° and 150°.

In the same way all angles corresponding to $\sin\theta = -\frac{1}{2}$ are given by

$$\theta = -30° + n \cdot 360° \quad \text{or} \quad -150° + n \cdot 360°;$$

or by

$$\theta = -\frac{\pi}{6} + 2n\pi \qquad \text{or} \quad -\frac{5\pi}{6} + 2n\pi.$$

All angles corresponding to $\sin\theta = 1$ are given by

$$\theta = 90° + n \cdot 360°; \qquad \text{or by} \quad \theta = \frac{\pi}{2} + 2n\pi.$$

Here there is only one basic angle, $\theta_1 = 90°$.

To determine all angles corresponding to the equation $\csc\theta = 2$, we note that this equation is the same as $\sin\theta = \frac{1}{2}$ and must have the same set of solutions which we have already found for the latter equation.

RULE: *All solutions of either of the equations*

$$\sin\theta = k \quad \text{or} \quad \csc\theta = k'$$

may be obtained by finding the basic angles (or the basic angle) and increasing each of the basic angles by $n \cdot 360°$, or by $2n\pi$ (radians).

(NOTE. The basic angles will lie in adjacent quadrants, either I, II or III, IV as in Figs. 26a, b. If k is not a possible value of the sine function, or k' of the cosecant function, there will be no solutions.)

Examples.

1. $\sin\theta = \frac{1}{3} = 0.3333+$. By interpolation from Table III, the basic angles, to the nearest minute, are $\theta_1 = 19°\ 28'$ and $\theta_2 = 160°\ 32'$.

Therefore all values of θ are given by

$$\theta = 19^\circ\ 28' + n \cdot 360^\circ \quad \text{or} \quad 160^\circ\ 32' + n \cdot 360^\circ.$$

2. $\csc \theta = -3$. From Table III we take the angles corresponding to $\csc \theta = +3$ and change their signs as explained in §23. We obtain $\theta_1 = -19^\circ\ 28'$ and $\theta_2 = -160^\circ\ 32'$. All values of θ are given by

$$\theta = -19^\circ\ 28' + n \cdot 360^\circ \quad \text{or} \quad -160^\circ\ 32' + n \cdot 360^\circ.$$

We next consider the equation $\cos \theta = k$, where k is any possible value of the cosine function, $-1 \leqq k \leqq 1$.

To illustrate, we use the equations below.

Given: $\cos \theta = \frac{1}{2}$; $\cos \theta = -\frac{1}{2}$; $\cos \theta = -1$.

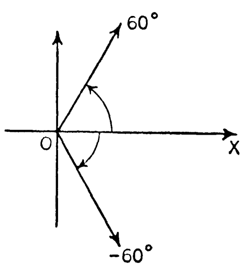


FIG. 27a

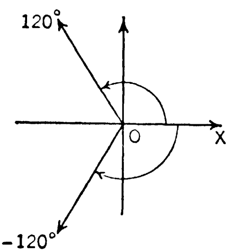


FIG. 27b

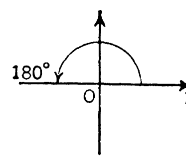


FIG. 27c

Quadrants:	I, IV;	II, III;	
Basic angles:	$+60^\circ, -60^\circ$;	$+120^\circ, -120^\circ$;	180°.
All angles:	$\pm 60^\circ + n \cdot 360^\circ$;	$\pm 120^\circ + n \cdot 360^\circ$;	$180^\circ + n \cdot 360$.

We see that the rule for finding all solutions of the equation $\sin \theta = k$ applies also to the equation $\cos \theta = k$ and to the equation $\sec \theta = k'$.

The basic angles, when there are two, again lie in adjacent quadrants; in quadrants I, IV if k is positive and in quadrants II, III if k is negative.

The same rule applies to the equations $\tan \theta = k$ and $\cot \theta = k'$.

In these cases the basic angles lie in opposite quadrants unless they are quadrantal angles; they lie in quadrants I, III if k is positive and in quadrants II, IV if k is negative.

35. EXERCISES 13

Obtain all solutions of the following equations. Give exact values, or to the nearest minute.

1. $\sin\theta = \frac{1}{\sqrt{2}}$.
2. $\sin\theta = -\frac{\sqrt{3}}{2}$.
3. $\cos\theta = \frac{\sqrt{3}}{2}$.
4. $\cos\theta = 1$.
5. $\csc\theta = 2$.
6. $\csc\theta = \frac{2}{\sqrt{3}}$.
7. $\sec\theta = \sqrt{2}$.
8. $\sec\theta = -2$.
9. $\tan\theta = 1$.
10. $\tan\theta = -\sqrt{3}$.
11. $\cot\theta = -1$.
12. $\cot\theta = \sqrt{3}$.
13. $\sin\theta = 0.2991$.
14. $\sin\theta = -0.2991$.
15. $\tan\theta = 0.6200$.
16. $\cot\theta = -0.6200$.
17. $\sec\theta = -1.8979$.
18. $\csc\theta = 1.8979$.

36. The inverse function notation.

It is often desirable to refer to an angle through the value of one of its functions. If we know that $\tan\alpha = 2$ we can say "α is an angle whose tangent is 2." If a roadway rises 6 feet in a horizontal distance of 100 feet, we can say that the road slopes upward at an angle whose tangent is 0.06.

The statement "*α is an angle whose tangent is 2*" is represented in mathematical shorthand by one of the forms

$$\alpha = \textit{arc tan } 2 \quad \text{or} \quad \alpha = \tan^{-1} 2.$$

Either of these is a short way of writing the quoted statement. It should be noted that the symbol "$\tan^{-1}\alpha$" is not the same as $(\tan\alpha)^{-1} = \frac{1}{\tan\alpha}$.

The symbols are read

"***arc tangent 2***" or "***inverse tangent 2***"

respectively. Either one represents the whole set of angles satisfying the equation ***tan*** $\alpha = 2$.

In general, in place of

$$\begin{array}{lll} \tan\alpha = a & \text{we write} & \alpha = \text{arc tan } a \quad \text{or} \quad \alpha = \tan^{-1} a; \\ \sin\alpha = a & \text{"} \quad \text{"} & \alpha = \text{arc sin } a \quad \text{or} \quad \alpha = \sin^{-1} a; \\ \sec\alpha = a & \text{"} \quad \text{"} & \alpha = \text{arc sec } a \quad \text{or} \quad \alpha = \sec^{-1} a; \end{array}$$

and corresponding equations for the other functions.

As we have seen, there is an unlimited number of such angles, consisting of the two basic angles (or the one basic angle) and all angles coterminal with them.

For definiteness, we single out one angle of this whole set and call it *the principal angle.*

DEFINITION. *The principal angle corresponding to a given value of a trigonometric function is the* **numerically** *smaller one of the two basic angles when these angles are unequal.*

When the basic angles are numerically equal but of opposite sign, the ***principal angle*** *is the* ***positive*** *basic angle.*

The basic angles are numerically equal for the inverse functions arc cos a and arc sec a. They are unequal for the other four inverse functions.

Examples.

Inverse function	*Basic angles*		*Princ. angle*
$\alpha = \text{arc sin } \frac{1}{2}$;	$\alpha_1 = 30°$,	$\alpha_2 = 150°$;	$\alpha_1 = 30°$.
$\alpha = \text{arc tan } (-1)$;	$\alpha_1 = -45°$,	$\alpha_2 = +135°$;	$\alpha_1 = -45°$.
$\alpha = \text{arc sec } (-2)$;	$\alpha_1 = 120°$,	$\alpha_2 = -120°$;	$\alpha_1 = 120°$.
$\alpha = \cos^{-1} 0.7402$;	$\alpha_1 = 42° 15'$,	$\alpha_2 = -42° 15'$;	$\alpha_1 = 42° 15'$.

Notation for the principal angle.

To indicate the principal angle we capitalize the first letter of the symbol for the inverse function. Thus:

$$\text{p.v. of arc sin } a = \text{Arc sin } a; \quad \text{p.v. of } \sin^{-1} a = \text{Sin}^{-1} a;$$
$$\text{p.v. of arc tan } a = \text{Arc tan } a; \quad \text{p.v. of } \tan^{-1} a = \text{Tan}^{-1} a.$$

EXERCISES 14

State the basic angles and the principal angle. Give exact answers when possible, otherwise to the nearest minute.

1. arc sin $\frac{\sqrt{3}}{2}$.
2. arc cos $(\frac{1}{2})$.
3. $\tan^{-1} \frac{1}{\sqrt{3}}$.
4. $\sec^{-1} \frac{2}{\sqrt{3}}$.
5. arc cot (-1).
6. $\sin^{-1} (-\frac{1}{2})$.
7. $\tan^{-1} (-2)$.
8. arc sec 3.
9. $\cos^{-1} 0.25$.
10. arc csc (-2.5).
11. arc sin $(\frac{2}{3})$.
12. $\tan^{-1} (\frac{8}{7})$.
13. arc cos $(-\frac{2}{3})$.
14. $\tan^{-1} (\frac{5}{3})$.
15. $\sec^{-1} (-\frac{5}{3})$.

State the exact value, or to the nearest minute.

16. Arc sin 0.3076.
17. Arc sin $(-\frac{2}{3})$.
18. Sin^{-1} 0.9498.
19. Arc cos $(\frac{2}{3})$.
20. Arc cos $(-\frac{3}{4})$.
21. Cos^{-1} 0.1570.
22. Arc tan 1.8000.
23. $\text{Tan}^{-1} (-1.8000)$.
24. Arc cot 2.
25. $\text{Cot}^{-1} (0.5400)$.
26. Arc sec 2.0500.
27. $\text{Sec}^{-1} (-\frac{4}{3})$.
28. Arc sec (-1).
29. Arc csc 1.2150.
30. $\text{Csc}^{-1} (-\frac{6}{5})$.

37. Variations of the problems discussed in §34.

Example 1.

Obtain all solutions of the equation $\sin 2x = -\frac{1}{2}$.

Solution. Let $\theta = 2x$. We have to solve $\sin\theta = -\frac{1}{2}$. All solutions are given by

$$\theta = n\cdot 360^\circ - 30^\circ \quad\text{and}\quad \theta = n\cdot 360^\circ - 150^\circ.$$
$$2x = n\cdot 360^\circ - 30^\circ \quad\text{and}\quad 2x = n\cdot 360^\circ - 150^\circ.$$

Therefore $x = n\cdot 180^\circ - 15^\circ$ and $x = n\cdot 180^\circ - 75^\circ$.

Let the student examine these values of angle x when $n = 0$, $n = \pm 1$, $n = \pm 2$, $n = \pm 3$, $n = \pm 4$.

Example 2.

Obtain all solutions of the equation $\tan 3x = 1$.

Solution. Let $\theta = 3x$. We have to solve $\tan\theta = 1$.

Therefore $\theta = n\cdot 360^\circ + 45^\circ$ and $\theta = n\cdot 360^\circ - 135^\circ$;

$$3x = n\cdot 360^\circ + 45^\circ \quad\text{and}\quad 3x = n\cdot 360^\circ - 135^\circ;$$
$$x = n\cdot 120^\circ + 15^\circ \quad\text{and}\quad x = n\cdot 120^\circ - 45^\circ.$$

Examine these answers when $n = 0, \pm 1, \pm 2, \pm 3, \pm 4$. Check some of them by substituting in the original equation.

Example 3.

Solve: $\tan(3x - 60^\circ) = 1$.

Solution. Let $\theta = 3x - 60^\circ$. As in Example 2,

$$\theta = n\cdot 360^\circ + 45^\circ \quad\text{and}\quad \theta = n\cdot 360^\circ - 135^\circ.$$
$$3x - 60^\circ = n\cdot 360^\circ + 45^\circ \quad\text{and}\quad 3x - 60^\circ = n\cdot 360^\circ - 135^\circ.$$
$$3x = n\cdot 360^\circ + 105^\circ \quad\text{and}\quad 3x = n\cdot 360^\circ - 75^\circ.$$
$$x = n\cdot 120^\circ + 35^\circ \quad\text{and}\quad x = n\cdot 120^\circ - 25^\circ.$$

Check some of these answers.

Example 4.

Solve: $\sec(\frac{5}{2}x - 30^\circ) = -3$.

Solution. Let $\theta = \frac{5}{2}x - 30^\circ$. Solve $\sec\theta = -3$.

Basic angles: $\theta_1 = 109^\circ\, 28'$ and $\theta_2 = -109^\circ\, 28'$.

All values of θ:

$$\theta = n\cdot 360^\circ + 109^\circ\, 28' \quad\text{and}\quad \theta = n\cdot 360^\circ - 109^\circ\, 28'.$$
$$\tfrac{5}{2}x - 30^\circ = n\cdot 360^\circ + 109^\circ\, 28' \quad\text{and}\quad \tfrac{5}{2}x - 30^\circ = n\cdot 360^\circ - 109^\circ\, 28'.$$
$$\tfrac{5}{2}x = n\cdot 360^\circ + 139^\circ\, 28' \quad\text{and}\quad \tfrac{5}{2}x = n\cdot 360^\circ - 79^\circ\, 28'.$$
$$x = n\cdot 144^\circ + 55^\circ\, 47' \quad\text{and}\quad x = n\cdot 144^\circ - 31^\circ\, 47'.$$

Check some of these answers.

To illustrate the use of the inverse function notation we again solve Examples 3 and 4.

Example 3.

Solve: $\tan(3x - 60°) = 1.$

Solution. $3x - 60° = \text{arc tan } 1.$

$$3x = \text{arc tan } 1 + 60°.$$

$$x = \tfrac{1}{3} \text{ arc tan } 1 + 20°.$$

We can now insert the values of arc tan 1 or leave the answer as it stands.

Example 4.

Solve: $\sec(\tfrac{5}{2}x - 30°) = -3.$

Solution. $\tfrac{5}{2}x - 30° = \sec^{-1}(-3).$

$$\tfrac{5}{2}x = \sec^{-1}(-3) + 30°.$$

$$x = \tfrac{2}{5}\sec^{-1}(-3) + 12°.$$

EXERCISE 15

Obtain all solutions of the following equations.

1. $\sin(2x - 30°) = \tfrac{1}{2}.$
2. $\sin(3x + 60°) = \dfrac{1}{\sqrt{2}}.$
3. $\cos(5x - 120°) = -\dfrac{1}{\sqrt{2}}.$
4. $\tan(\tfrac{4}{3}x + 30°) = -1.$
5. $\sec(8x + 40°) = -2.$
6. $\cot(\tfrac{3}{2}\alpha + 15°) = 2.$
7. $\cos(\tfrac{2}{3}\beta - 20°) = 0.2991.$
8. $\tan(\tfrac{1}{3}\beta + 30°) = -0.6200.$
9. $\sec(4\alpha + 80°) = -1.$
10. $\cot(80° - 4\alpha) = 0.$

38. Given one function of an angle, to find the other functions.

Example 1.

$\sin x = \tfrac{1}{2}.$ Find the other functions.

Take ordinate $= 1$ and distance $= 2$; then abscissa $= \pm\sqrt{3}$ (figure). Then

$$\cos x = \pm\frac{\sqrt{3}}{2}, \quad \tan x = \pm\frac{1}{\sqrt{3}},$$

$$\cot x = \pm\sqrt{3}, \quad \sec x = \pm\frac{2}{\sqrt{3}},$$

$$\csc x = 2.$$

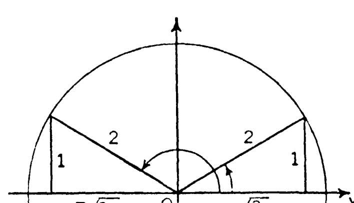


FIG. 28

We have found *two values* for each function except $\csc x$, which is the reciprocal of the given function. Similar results will be found in general. Note that the basic angles have the terminal lines shown in the figure.

Example 2.

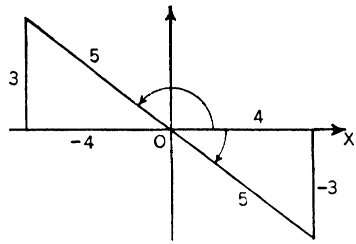


Fig. 29

$$\tan x = -\frac{3}{4}\left(= \frac{-3}{+4} \text{ or } \frac{+3}{-4}\right).$$

The two possible positions of the terminal line are shown in the figure.

Hence $\sin x = \pm \frac{3}{5}$, $\cos x = \mp \frac{4}{5}$,
$\cot x = -\frac{4}{3}$, $\csc x = \pm \frac{5}{3}$,
$\sec x = \mp \frac{5}{4}$.

Example 3.

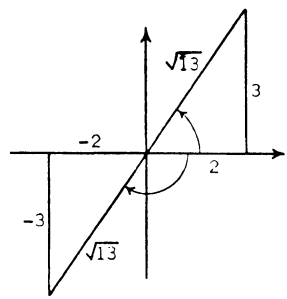


Fig. 30

$$\cot x = \frac{2}{3}\left(= \frac{+2}{+3} \text{ or } \frac{-2}{-3}\right).$$

Then (figure),

$$\sin x = \pm \frac{3}{\sqrt{13}}, \quad \cos x = \pm \frac{2}{\sqrt{13}},$$
$$\tan x = \tfrac{3}{2},$$
$$\csc x = \pm \frac{\sqrt{13}}{3}, \quad \sec x = \pm \frac{\sqrt{13}}{2}.$$

Example 4.

$$\sin x = \frac{h}{k}.$$

Ordinate $= h$; distance $= k$; hence abscissa $= \pm \sqrt{k^2 - h^2}$.

Then $\quad \cos x = \pm \dfrac{\sqrt{k^2 - h^2}}{k}, \quad \tan x = \pm \dfrac{h}{\sqrt{k^2 - h^2}}, \quad$ etc.

39. EXERCISES 16

Find the other functions, given that

1. $\sin x = -\frac{2}{3}$.
2. $\cos x = \frac{2}{3}$.
3. $\tan x = -\frac{3}{4}$.
4. $\sec x = 5$.
5. $\cot x = -\sqrt{3}$.
6. $\csc x = -\frac{13}{5}$.
7. $\sec x = -\frac{41}{9}$.
8. $\cot x = -0.8$.
9. $\sin x = -0.8$.
10. $\cos \theta = \sqrt{a}$.
11. $\csc \theta = -m$.
12. $\tan \theta = \dfrac{a}{b}$.
13. $\sin \varphi = 1 + h$.
14. $\cot \varphi = \sqrt{a - 1}$.
15. $\cos \varphi = \dfrac{2a}{a^2 + 1}$.

16. State for what values, if any, of the literal quantities in exercises 10–15, the given equations are impossible.

40. To express all the functions in terms of one of them.

1. Express all the functions in terms of the cosine.

We have

$$\cos x = \frac{\cos x}{1} = \frac{\text{abscissa}}{\text{distance}}.$$

Hence let abscissa $= \cos x$ and distance $= 1$.

Then ordinate $= \pm\sqrt{\text{dist.}^2 - \text{absc.}^2} = \pm\sqrt{1 - \cos^2 x}$.

The figure shows this graphically when $\cos x$ is positive.

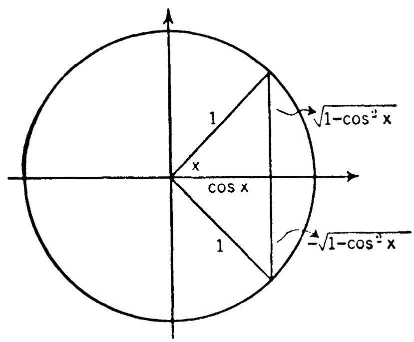


Fig. 31

Taking into account both values of the ordinate, we have

$$\sin x = \pm\sqrt{1 - \cos^2 x};$$

$$\tan x = \pm\frac{\sqrt{1 - \cos^2 x}}{\cos x};$$

$$\cot x = \pm\frac{\cos x}{\sqrt{1 - \cos^2 x}};$$

$$\csc x = \pm\frac{1}{\sqrt{1 - \cos^2 x}};$$

$$\sec x = \frac{1}{\cos x}.$$

Exercise 1. Draw a figure for the case when $\cos x$ is negative.

Exercise 2. Obtain the same equations directly from the formulas of Group A.

2. Express all the functions in terms of the cotangent.

$$\cot x = \frac{\cot x}{1} = \frac{-\cot x}{-1} = \frac{\text{abscissa}}{\text{ordinate}}.$$

Hence let abscissa $= \cot x$ and ordinate $= 1$.
or let abscissa $= -\cot x$ and ordinate $= -1$.
In either case, distance $= +\sqrt{1+\cot^2 x}$. (See figure, where we assume $\cot x > 0$.)

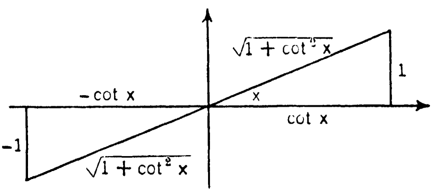


FIG. 32

Hence

$$\sin x = \pm \frac{1}{\sqrt{1+\cot^2 x}},$$

$$\cos x = \pm \frac{\cot x}{\sqrt{1+\cot^2 x}}, \quad \text{etc.}$$

EXERCISES 17

1. By taking each of the functions in turn, and proceeding as above, obtain the results shown in the following table. The given function and its reciprocal are uniquely determined; the other four functions are ambiguous in sign.

	$\sin x$.	$\cos x$.	$\tan x$.	$\cot x$.	$\sec x$.	$\csc x$.
$\sin x$	....	$\pm\sqrt{1-\cos^2 x}$	$\frac{\tan x}{\pm\sqrt{1+\tan^2 x}}$	$\frac{1}{\pm\sqrt{1+\cot^2 x}}$	$\frac{\pm\sqrt{\sec^2 x-1}}{\sec x}$	$\frac{1}{\csc x}$
$\cos x$	$\pm\sqrt{1-\sin^2 x}$	....	$\frac{1}{\pm\sqrt{1+\tan^2 x}}$	$\frac{\cot x}{\pm\sqrt{1+\cot^2 x}}$	$\frac{1}{\sec x}$	$\frac{\pm\sqrt{\csc^2 x-1}}{\csc x}$
$\tan x$	$\frac{\sin x}{\pm\sqrt{1-\sin^2 x}}$	$\frac{\pm\sqrt{1-\cos^2 x}}{\cos x}$	....	$\frac{1}{\cot x}$	$\pm\sqrt{\sec^2 x-1}$	$\frac{1}{\pm\sqrt{\csc^2 x-1}}$
$\cot x$	$\frac{\pm\sqrt{1-\sin^2 x}}{\sin x}$	$\frac{\cos x}{\pm\sqrt{1-\cos^2 x}}$	$\frac{1}{\tan x}$	....	$\frac{1}{\pm\sqrt{\sec^2 x-1}}$	$\pm\sqrt{\csc^2 x-1}$
$\sec x$	$\frac{1}{\pm\sqrt{1-\sin^2 x}}$	$\frac{1}{\cos x}$	$\pm\sqrt{1+\tan^2 x}$	$\frac{\pm\sqrt{1+\cot^2 x}}{\cot x}$	....	$\frac{\csc x}{\pm\sqrt{\csc^2 x-1}}$
$\csc x$	$\frac{1}{\sin x}$	$\frac{1}{\pm\sqrt{1-\cos^2 x}}$	$\frac{\pm\sqrt{1+\tan^2 x}}{\tan x}$	$\pm\sqrt{1+\cot^2 x}$	$\frac{\sec x}{\pm\sqrt{\sec^2 x-1}}$	....

2. Express $\cos^2 x - \sin^2 x$ in terms of $\tan x$.
3. Express $\cot x \csc x + \csc^2 x$ in terms of $\sin x$.
4. Express $\sin^2 x \tan x$ in terms of $\cot x$.
5. Express $\dfrac{1}{1 + \sin \theta} + \dfrac{1}{1 - \sin \theta}$ in terms of $\csc \theta$.
6. Express $\dfrac{\cos \theta}{1 - \tan \theta}$ in terms of $\sec \theta$.

41. Trigonometric equations.

A trigonometric equation is an equation which involves one or more trigonometric functions of one or more angles and which is not an identity. Thus:

$$\sin^2 x + \cos x = 1; \quad \tan \theta + \sec \theta = 3; \quad \cot \alpha \csc \alpha = 2.$$

To find the values of the angle which satisfy such an equation, it is usually best to use a method adapted to the case in hand. We give here one general rule, which covers a considerable variety of cases.

RULE: *To solve a trigonometric equation, express all its terms by means of a single function of the unknown angle; solve as an algebraic equation, considering this function as unknown; find the angles corresponding to the values of the function so obtained. Check all answers by substitution.*

In this reduction we usually shall need one or more of the identities of Group A.

Example 1.

$\tan^2 x + \tan x = 2$. Solve for x.

This is a quadratic equation* with $\tan x$ as the unknown. Let $y = \tan x$.

$$y^2 + y = 2; \quad y^2 + y - 2 = 0; \quad (y - 1)(y + 2) = 0.$$

Therefore $y = 1$ or $y = -2$; $\tan x = 1$ or $\tan x = -2$.
$\tan x = 1$: $x = \tan^{-1} 1 = 45° + n \cdot 360°$ or $-135° + n \cdot 360°$.
$\tan x = -2$: $x = \tan^{-1}(-2) = -63° 26' + n \cdot 360°$ or $116° 34' + n \cdot 360°$.

* We recall the quadratic formula for use when one can not factor by inspection.

If $ay^2 + by + c = 0$, then $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Check. All these angles check because for the first set $\tan x = 1$ and for the second set $\tan x = -2$. Both of these values of $\tan x$ check in the original equation.

Example 2.

$\sin^2 x + \cos x = 1.$

Substitute $\sin^2 x = 1 - \cos^2 x$; $1 - \cos^2 x + \cos x = 1$;

$\cos^2 x - \cos x = 0$; $\cos x\,(\cos x - 1) = 0$.

Therefore $\cos x = 0$ or $\cos x = 1$.

$$\cos x = 0\colon\ x = \text{arc}\cos 0 = \frac{\pi}{2} + 2n\pi \quad \text{or} \quad -\frac{\pi}{2} + 2n\pi.$$

$$\cos x = 1\colon\ x = \text{arc}\cos 1 = 0 + 2n\pi = 2n\pi.$$

Check. For the first set of angles, $\cos x = 0$ and $\sin x = \pm 1$. For the second set of angles, $\cos x = 1$ and $\sin x = 0$. All check.

Example 3.

$\tan\theta + \sec\theta = 3.$

Transpose and square:

$$\sec\theta = 3 - \tan\theta;\ \sec^2\theta = 9 - 6\tan\theta + \tan^2\theta.$$

Substitute $\sec^2\theta = 1 + \tan^2\theta$ and collect terms:

$$6\tan\theta = 8;\ \tan\theta = \tfrac{4}{3}.$$

$$\theta = \text{arc}\tan\tfrac{4}{3} = 53°\,8' + n\cdot 360° \quad \text{or} \quad -126°\,52' + n\cdot 360°.$$

Check. The process of squaring the members of an equation usually introduces extraneous solutions. Thus: $2x = 1$ has one solution; $4x^2 = 1$ has two solutions. In our example the angles of the second set, $-126°\,52' + 2n\pi$ do not satisfy the given equation. For these angles $\tan\theta = \frac{4}{3}$ and $\sec\theta = -\frac{5}{3}$.

Example 4.

$2\cos\theta + \sin\theta = 2.$

Transpose and square:

$$2\cos\theta = 2 - \sin\theta;\ 4\cos^2\theta = 4 - 4\sin\theta + \sin^2\theta.$$

Substitute $\cos^2\theta = 1 - \sin^2\theta$ and simplify:

$$5\sin^2\theta - 4\sin\theta = 0;\ \sin\theta\,(5\sin\theta - 4) = 0;$$

$$\sin\theta = 0 \quad \text{or} \quad \sin\theta = \tfrac{4}{5} = 0.8.$$

$$\theta = \text{arc}\sin 0 = 0° + n\cdot 360° \quad \text{or} \quad 180° + n\cdot 360°.$$

$$\theta = \text{arc}\sin 0.8 = 53°\,8' + n\cdot 360° \quad \text{or} \quad 126°\,42' + n\cdot 360°.$$

Check. The values of θ which check are

$$\theta = n\cdot 360° \quad \text{and} \quad \theta = 53°\,8' + n\cdot 360°.$$

The other values must be discarded.

EXERCISES 18

Solve for the unknown angle.

1. $2 \sin^2 x - \sin x = 0.$
2. $4 \sec^2 x = 3.$
3. $2 \sec x = 1 + \tan^2 x.$
4. $2 \cos^2 x = 1 - \cos x.$
5. $\tan^2 \theta + \sec^2 \theta = 3.$
6. $\csc^2 \theta + \cot^2 \theta = 5.$
7. $\tan \theta = 2 \sin \theta.$
8. $\tan^2 \theta + \sec \theta = 1.$
9. $2 \sin^2 \theta - 2 \cos^2 \theta = 1.$
10. $1 + \tan^2 \theta = 2 \tan \theta.$
11. $1 - \tan^2 \alpha = 2 \tan \alpha.$
12. $2 \cos^2 \alpha - 2 \sin^2 \alpha = \sqrt{2}.$
13. $\tan \alpha - \sec \alpha = 3.$
14. $\sec \alpha = 1 + \tan \alpha.$
15. $2 \sin \theta + \cos \theta = 2.$
16. $\sin \theta - 2 \cos \theta = 2.$
17. $\tan \theta + 2 = 3 \cot \theta.$
18. $2 \sec \theta = 3 + 2 \cos \theta.$
19. $3 \sin x + 4 \cos x = 5.$
20. $5 \sin x + 4 \cos x = 4.$

CHAPTER

IV

LOGARITHMIC SOLUTION OF RIGHT TRIANGLES.* APPLICATIONS.

PART I. SOLUTION OF TRIANGLES

42. Remarks on numerical computations.

Suppose a given quantity has the exact numerical measure N. This might be N feet, N pounds, N bushels, and so on. Let $N = 20673$, a 5-digit number.

To express this number with 4-digit accuracy, or, *to 4 significant digits,* we keep the first three digits, 2, 0, 6, and *round off* the 73 to 70. To express N to 3 significant digits, we keep the first two digits, and round off 673 to 700.

(a) To four significant digits: $N = 20670$.
(b) To three " " : $N = 20700$.

We follow the same plan for decimal numbers. If $N = 0.020673$, then

(c) to four significant digits, $N = 0.02067$;
(d) to three " " , $N = 0.0207$.

But observe in this case that *final zeros following the decimal point are omitted.*

* For those who have not studied logarithms, a full discussion of the theory and use of logarithms and of the use of tables is given in Appendix B.

The statements (a) and (b) without any more exact information about N, mean respectively that:

the exact value of N lies between 20665 and 20675;
" " " " " " " 20650 and 20750.

Accuracy obtainable by the use of tables.

A theoretical study of this question is beyond the scope of this book. We briefly summarize the results.

1. Tables of logarithms of numbers. In general, 4-place tables of $\log N$ will yield N to not more than 4 significant digits; 5-place tables of $\log N$ will yield N to not more than 5 significant digits.

2. Tables of natural or logarithmic trigonometric functions. In general, 4-place tables will yield angles to the nearest minute, and 5-place tables will yield angles to the nearest tenth of a minute. Where the tabular differences are large, the accuracy will be somewhat greater; where the tabular differences are small, the accuracy will be less.

3. Interpolations should not be carried out more than one place beyond the number of places in the table. Then round off the result.

Examples.

1. From Table I, $\log 30.23 = 1.4800 + 0.3(.0014)$. But $0.3(.0014) = 0.00042 = 0.0004$ (rounded off). Therefore $\log 30.23 = 1.4800 + 0.0004 = 1.4804$. Here the digit 2 is not significant because 0.0014 is given only to the fourth decimal place. We should do only as much work as is necessary to get the nearest digit in the fourth decimal place.

2. From Table III, if $\cos x = 0.8650$, $x = 30°\,0' + \frac{10}{14}(10')$. We might calculate $\frac{10}{14}(10') = 7.14+'$. But this is useless refinement because our 4-place table will yield angles only to the nearest minute. So we divide out to get $7.1'$ and then shorten to $7'$. Then $x = 30°\,7'$.

As the student becomes familiar with the tables he will see that, while the statements made above are true in general, at some places in the table the accuracy is greater than that stated, and much less at other places. For example, $\cos x = 0.9998$ will *not* determine x to the nearest minute.

43. Logarithmic solution of right triangles.

As explained in §13, the trigonometric functions are utilized to solve right triangles. This problem may be conveniently discussed under four cases, according to the nature of the given parts.

1. Given the hypotenuse and an acute angle.
2. Given a side and an acute angle.
3. Given the hypotenuse and a side.
4. Given the two sides.

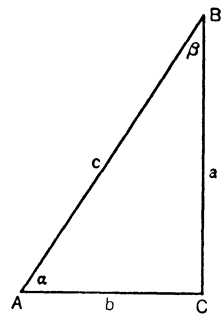


FIG. 33

The formulas to be used are:

$$\sin\alpha = \cos\beta = \frac{a}{c}. \qquad \cos\alpha = \sin\beta = \frac{b}{c}.$$

$$\tan\alpha = \cot\beta = \frac{a}{b}. \qquad \cot\alpha = \tan\beta = \frac{b}{a}.$$

$$\alpha + \beta = 90°. \qquad c^2 = a^2 + b^2.$$

To calculate an unknown part when two parts of the triangle are given select that equation which contains the unknown part and the two given parts.

A modified form of the last equation is commonly used as a check; its use in finding the unknown parts should be avoided.

Case 1. Given the hypotenuse and an angle, as c and α.

Formulas for calculating a, b, β.

Angle β: $\quad \beta = 90° - \alpha.$

Side a: $\quad \frac{a}{c} = \sin\alpha;\ a = c\sin\alpha;\ \log a = \log c + \log\sin\alpha.$

Side b: $\quad \frac{b}{c} = \cos\alpha;\ b = c\cos\alpha;\ \log b = \log c + \log\cos\alpha.$

Check.

$$b^2 = c^2 - a^2 = (c-a)(c+a);$$
$$\log b = \tfrac{1}{2}[\log(c-a) + \log(c+a)].$$
$$a^2 = c^2 - b^2 = (c-b)(c+b);$$
$$\log a = \tfrac{1}{2}[\log(c-b) + \log(c+b)].$$

Use that check formula which contains the larger of the two differences $c - a$, $c - b$.

Example 1.

Four-place tables.

Given $\mathbf{c = 24.37}$, $\boldsymbol{\alpha} = \mathbf{32° 12'}$. Find $\boldsymbol{a, b, \beta}$.

Angle β.	*Side a.*	*Side b.*
$\beta = 90° - \alpha$	$\log c = 1.3869$	$\log c = 1.3869$
$= \mathbf{57° 48'}$.	$\log \sin \alpha = \underline{9.7266{-}10}$	$\log \cos \alpha = \underline{9.9274{-}10}$
	$\log \boldsymbol{a} = 1.1135$	$\log b = 1.3143$
	$\boldsymbol{a} = \mathbf{12.99}$	$\boldsymbol{b} = \mathbf{20.61}$

Check.

$c - a = 11.38$	$\log (c - a) = 1.0561$	$\frac{1}{2}$ sum $= 1.3142$
$c + a = 37.36$	$\log (c + a) = \underline{1.5723}$	$\log b = 1.3143.$
	sum $= 2.6284$	

Example 2.

Five-place tables.

Given $\mathbf{c = 24.373}$, $\boldsymbol{\alpha} = \mathbf{32° 12.7'}$. Find $\boldsymbol{a, b, \beta}$.

Angle β	*Side a*	*Side b.*
$\beta = 90° - \alpha.$	$\log c = 1.38691$	$\log c = 1.38691$
$= \mathbf{57° 47.3'}$.	$\log \sin \alpha = \underline{9.72677 - 10}$	$\log \cos \alpha = \underline{9.92741 - 10}$
	$\log a = 1.11368$	$\log b = 1.31432$
	$\boldsymbol{a} = \mathbf{12.992}$	$\boldsymbol{b} = \mathbf{20.621}$

Check.

$c - a = 11.381$	$\log (c - a) = 1.05618$	$\frac{1}{2}$ sum $= 1.31432.$
$c + a = 37.365$	$\log (c + a) = \underline{1.57246}$	$\log b = 1.31432.$
	sum $= 2.62864$	

Case 2. Given a side and an angle, as a and α.

Formulas for calculating b, c, β.

Angle β: $\qquad \beta = 90° - \alpha.$

Side b: $\quad \dfrac{b}{a} = \cot \alpha;\ b = a \cot \alpha;$

$$\log b = \log a + \log \cot \alpha.$$

Hyp. c: $\quad \dfrac{c}{a} = \dfrac{1}{\sin \alpha};\ c = \dfrac{a}{\sin \alpha};$

$$\log c = \log a - \log \sin \alpha.$$

Check. As in Case 1.

Example 1.

Four-place tables.

Given $\boldsymbol{a} = \mathbf{27.32}$, $\alpha = \mathbf{37° \, 33'}$. Find $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\beta}$.

Angle β.	*Side b.*	*Hyp. c.*
$\beta = 90° - \alpha.$	$\log a = 1.4365$	$\log a = 1.4365$
$= \mathbf{52° \, 27'}.$	$\log \cot \alpha = \underline{0.1142}$	$\log \sin \alpha = \underline{9.7849 - 10}$
	$\log b = 1.5507$	$\log c = 1.6516$
	$\boldsymbol{b} = \mathbf{35.53}.$	$\boldsymbol{c} = \mathbf{44.83}.$

Check.

$c - a = 17.51$	$\log (c - a) = 1.2432$	$\frac{1}{2}$ sum $= 1.5507$
$c + a = 72.15$	$\log (c + a) = 1.8582$	$\log b = 1.5507.$
	sum $= 3.1014$	

Example 2.

Five-place tables.

Given $\boldsymbol{a} = \mathbf{27.326}$, $\alpha = \mathbf{37° \, 33.8'}$. Find $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\beta}$.

Angle β.	*Side b.*	*Hyp. c.*
$\beta = 90° - \alpha$	$\log a = 1.43658$	$\log a = 1.43658$
$= \mathbf{52° \, 26.2'}.$	$\log \cot \alpha = \underline{0.11402}$	$\log \sin \alpha = \underline{9.78507 - 10}$
	$\log b = 1.55060$	$\log c = 1.65151$
	$\boldsymbol{b} = \mathbf{35.530}.$	$\boldsymbol{c} = \mathbf{44.824}.$

Check.

$c - a = 17.498$	$\log (c - a) = 1.24299$	$\frac{1}{2}$ sum $= 1.55061$
$c + a = 72.150$	$\log (c + a) = \underline{1.85824}$	$\log b = 1.55060$
	sum $= 3.10123$	

Case 3. Given the hypotenuse and a side, as c and a.

Formulas for calculating b, α, β.

Angle α: $\sin \alpha = \dfrac{a}{c}$; $\log \sin \alpha = \log a - \log c$.

Angle β: $\beta = 90° - \alpha$.

Side b: $b = c \cos \alpha$; $\log b = \log c + \log \cos \alpha$.

Check. As before.

A form for the computations may now be made out as in preceding examples.

Case 4. Given the two sides, a and b.

Formulas for finding c, α, β.

Angle α: $\tan \alpha = \frac{a}{b}$; $\log \tan \alpha = \log a - \log b$.

Angle β: $\beta = 90° - \alpha$.

Hyp. c: $c = \frac{a}{\sin \alpha}$; $\log c = \log a - \log \sin \alpha$.

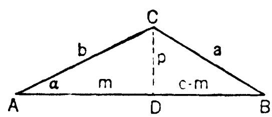
Check. As before.

Solution of oblique triangles by mean of right triangles.

In the oblique plane triangle ABC we designate the angles at A, B, C respectively by α, β, γ and the opposite sides by a, b, c.

Example 1.

Given $b = 12.55$, $c = 20.63$, $\alpha = 27° 24'$. Determine a, β, γ.



Draw CD perpendicular to AB and let $AD = m$. (Figure.) In right triangle CDA we know b and α and can solve for m and p. Then in right triangle CDB we have p and $c - m$ and can solve for a and β. Finally $\gamma = 180° - (\alpha + \beta)$.

Formulas.			*Check.*
$\triangle CDA$:	$m = b \cos \alpha$;	$p = b \sin \alpha$.	$p = m \tan \alpha$.
$\triangle CDB$:	$\tan \beta = p/(c - m)$;	$a = p/\sin \beta$.	$(c - m)^2 = (a + p)(a - p)$.

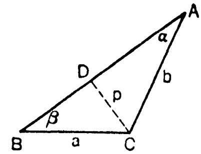
Solution.

$\log b$	$= 1.0986-10$	$\log b$	$= 1.0986-10$	$\log m$	$= 1.0469$
$\log \cos \alpha$	$= 9.9483-10$	$\log \sin \alpha$	$= 9.6630-10$	$\log \tan \alpha$	$= 9.7146-10$
$\log m$	$= 1.0469$	$\log p$	$= 0.7616$	sum	$= 0.7615$
				$\log p$	$= 0.7616$
m	$= 11.14$	p	$= 5.775$		
$c - m$	$= 9.49$				
$\log p$	$= 0.7616$	$\log p$	$= 0.7616$	$a + p$	$= 16.88$
$\log (c - m)$	$= 0.9773$	$\log \sin \beta$	$= 9.7158-10$	$a - p$	$= 5.34$
$\log \tan \beta$	$= 9.7843-10$	$\log a$	$= 1.0458$	$\log (a + p)$	$= 1.2274$
				$\log (a - p)$	$= 0.7275$
	$\beta = 31° 19'$		$a = 11.11.$	sum	$= 1.9549$
	$\alpha = 27° 24'$			$\frac{1}{2}$ sum	$= 0.9774$
sum	$58° 43'$	$\gamma = 180° - 58° 43' = 121° 17'.$			(ch.)

Example 2.

Given $a = 351.2$, $\alpha = 28° 20'$, $\beta = 35° 45'$. Find b, c, γ.

Draw CD perpendicular to AB. (Figure.)



Formulas.

$\triangle CDB$: $BD = a \cos \beta$; $p = a \sin \beta$.
$\triangle CDA$: $b = p/\sin \alpha$; $AD = p \cot \alpha$.
$c = BD + DA$; $\gamma = 180° - (\alpha + \beta)$.

The numerical solution is left as an exercise for the student.

44. EXERCISES 19

In Exercises 1–24 solve by 4-place logarithmic tables, including the check. In each case give answers to the limit of accuracy obtainable by the tables. Where the tabular differences are small, say less than 20, practice making interpolations mentally, without reference to the table of proportional parts.

1. $c = 57.56$; $\alpha = 64° 41'$.
2. $b = 24.61$; $\beta = 25° 19'$.
3. $c = 2738$; $\beta = 31° 7'$.
4. $a = 2344$; $\alpha = 58° 53'$.
5. $a = 1507$; $\alpha = 29° 31'$.
6. $c = 3058$; $\beta = 60° 29'$.
7. $b = .4332$; $\beta = 21° 33'$.
8. $c = .1179$; $\alpha = 68° 27'$.
9. $a = 3157$; $\beta = 36° 41'$.
10. $b = 2352$; $\alpha = 53° 19'$.
11. $b = .0267$; $\alpha = 73° 0'$.
12. $c = .0913$; $\beta = 17° 0'$.
13. $c = 919.9$; $\beta = 14° 52'$.
14. $a = 889.0$; $\alpha = 75° 8'$.
15. $a = .03562$; $\beta = 48° 42'$.
16. $b = .04055$; $\alpha = 41° 18'$.
17. $b = 24.61$; $c = 57.56$.
18. $a = .1097$; $b = .4332$.
19. $a = 3157$; $b = 2352$.
20. $a = 1507$; $c = 3058$.
21. $c = .0913$; $a = .0873$.
22. $c = 2738$; $b = 1415$.
23. $b = .04055$; $a = .03562$.
24. $b = 14247$; $a = 12758$.

In Exercises 25–40 use 5-place tables.

25. $a = 23.646$; $\alpha = 39° 0.8'$.
26. $b = 163.15$; $\alpha = 58° 35.3'$.
27. $c = 19124$; $\beta = 48° 9.3'$.
28. $c = 37.562$; $\beta = 50° 59.2'$.
29. $a = 267.15$; $\beta = 31° 24.7'$.
30. $b = .30854$; $c = .49267$.
31. $c = 481.67$; $a = 234.52$.
32. $a = .38408$; $\beta = 38° 46.6'$.
33. $b = 420.72$; $\alpha = 29° 8.2'$.
34. $b = 2081.5$; $a = 6832.4$.
35. $a = 32.567$; $b = 26.873$.
36. $c = 43205$; $\alpha = 41° 31.3'$.
37. $c = 42.223$; $\beta = 39° 31.7'$.
38. $a = 12000$; $b = 1500$.
39. $b = 32347$; $c = 43205$.
40. $c = 120.65$; $\beta = 7° 5.5'$.

Oblique plane triangles. Solve for the three parts not given. Use 4-place tables.

41. $b = 177$; $c = 217$; $\alpha = 60°$.
42. $a = 120$; $b = 210$; $\gamma = 58° 50'$.
43. $a = 160$; $c = 236$; $\beta = 56° 46'$.
44. $a = 800$; $\alpha = 60°$; $\beta = 50°$.
45. $c = 180$; $\alpha = 34° 45'$; $\beta = 86° 25'$.

PART II. PROBLEMS IN HEIGHTS AND DISTANCES

45. Angle of elevation; angle of depression.

Let O be a point from which the line of sight to a point A is elevated through an angle α, and the line of sight to point B is depressed through an angle β, both angles measured from the horizontal line OH.

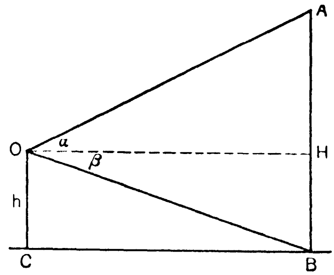


FIG. 34

Angle α is the ***angle of elevation*** of line OA, or of point A.

Angle β is the ***angle of depression*** of line OB, or of point B.

Let CB be drawn parallel to OH and let $h = CO$ be the height of point O above C.

If h, α, β are given, the lengths of all the lines in the figure can be calculated.

EXERCISES

1. Obtain the following equations: —

$$\begin{aligned}
\triangle\, OCB:&\quad CB = h\cot\beta;\quad OB = h\csc\beta.\\
\triangle\, OHB:&\quad OH = CB;\quad BH = h.\\
\triangle\, OHA:&\quad OH = CB = h\cot\beta;\\
&\quad HA = OH\tan\alpha = h\cot\beta\tan\alpha;\\
&\quad OA = OH\sec\alpha = h\cot\beta\sec\alpha.
\end{aligned}$$

2. Calculate the values of these quantities when

$$h = 250 \text{ feet};\quad \alpha = 35°;\quad \beta = 25°.$$

46. Width of a river.

To determine the width of a river, $w = AB$, a surveyor might set his transit at A, sight across to a well marked point B, turn off 90° into the line AC, and have a stake set at some convenient point C. Measure $AC = m$, and from C measure $\angle\, ACB = \alpha$.

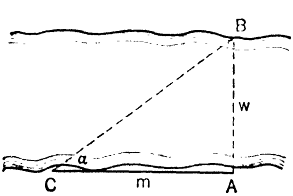


FIG. 35

Then from $\triangle\, ABC$ we have

$$\frac{w}{m} = \tan\alpha,\quad \text{or,}\quad w = m\tan\alpha.$$

EXERCISES

Calculate w when

	(a)	(b)	(c)	(d)
$m =$	227 ft.	129.5 ft.	663 ft.	387 ft.
$\alpha =$	$51° 43'$.	$31° 26'$.	$42° 17'$.	$19° 33'$.

NOTE. Logarithms should be used in these calculations. Check results roughly by measurement of figures drawn to scale.

47. Height of an inaccessible object.

To find h, the height of a hill, (Fig. 36), choose a point A on level ground and measure $\angle CAD = \alpha$, called "the angle of elevation." Then approach a measured distance m on level ground, to B; at B measure the angle of elevation β. Now α, β, and m are known; to calculate h.

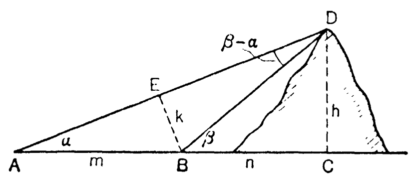


FIG. 36

First Solution. Let $BC = n$.

Then $\dfrac{n}{h} = \cot\beta$, and $\dfrac{m+n}{h} = \cot\alpha$.

Subtracting: $\dfrac{m}{h} = \cot\alpha - \cot\beta$; hence $h = \dfrac{m}{\cot\alpha - \cot\beta}$.

Second Solution. Let k be the length of the perpendicular from B on AD. Then we can calculate, in order, first k, next BD, and finally h.

From $\triangle ABE$: $k = m\sin\alpha$.

From $\triangle BED$: $BD = \dfrac{k}{\sin(\beta-\alpha)} = m\dfrac{\sin\alpha}{\sin(\beta-\alpha)}$.

From $\triangle BDC$: $h = BD\sin\beta = m\dfrac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$.

For logarithmic calculation this formula is much better than the preceding. It gives

$$\log h = \log m + \log\sin\alpha + \log\sin\beta + \operatorname{colog}\sin(\beta-\alpha).$$

EXERCISES

1. What does the second solution give when $\beta = 2\alpha$? Explain.

2. Use both formulas to find h when

	(a)	(b)	(c)	(d)
$\alpha =$	$20°$,	$15° 48'$,	$27° 33'$,	$32° 18.3'$.
$\beta =$	$25°$,	$22° 17'$,	$41° 07'$,	$43° 36.7'$.
$m =$	350 ft.	189.7 ft.	228.3 ft.	7447.6 ft.

Draw figures to scale and give the graphic solutions.

48. Height of an inaccessible object. Second method.

Let CD stand perpendicular to the horizontal plane MN. To determine the height CD or h.

From A measure $\angle\ \alpha$; if now we cannot approach C or recede from it on account of obstacles such as trees, or a river, or other barrier, lay off a measured distance $AB = m$, at right angles to AC; at B measure $\angle\ \beta$.

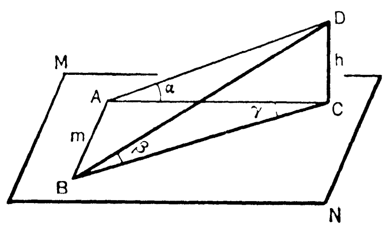


FIG. 37

Given m, α, β; to calculate h.

Solution. Let $\gamma = \angle\ ACB$. $\cos\gamma = AC \div BC$.

But $AC = h\cot\alpha$; $BC = h\cot\beta$.

$$\therefore \quad \cos\gamma = \frac{\cot\alpha}{\cot\beta},$$

from which γ may be found.

Knowing $\angle\ \gamma$ and m, we can calculate either AC or BC, and then h. Thus:

$$AC = m\cot\gamma; \quad h = AC\tan\alpha = m\cot\gamma\tan\alpha.$$

Our scheme for logarithmic calculation would be:

$$\log\cos\gamma = \log\cot\alpha - \log\cot\beta; \quad \gamma = ?$$

$$\log h = \log m + \log\cot\gamma + \log\tan\alpha; \quad h = ?$$

EXERCISES

1. Calculate h when $m = 1575$ feet; $\alpha = 32°$; $\beta = 19°$.
2. Calculate h when $m = 236.7$ feet; $\alpha = 58° 16'$; $\beta = 40° 34'$.

49. EXERCISES 20

1. A building 212 feet high casts a shadow 683 feet long. Find the angle of elevation of the sun.

2. If an airplane glides downward at an angle of 15° with the horizontal, how many feet will it descend while traveling a distance of 20,000 feet?

3. The Leaning Tower of Pisa is 179 feet high and is out of plumb 16.5 feet. At what angle does it lean from the vertical?

4. A pole 17.25 feet long casts a shadow on level ground 25.75 feet long. What is the angle of elevation of the sun?

5. From a battery at the top of a cliff 1537 feet above sea level the angle of depression of a ship is 15° 10′. Find the horizontal distance to the ship.

6. A level road makes an angle of 5° with the horizontal. How many feet will an automobile rise in traveling 5 miles along the road?

7. Two towers stand on level ground and are 2537 feet apart. From a point on the ground midway between the towers the angle of elevation of one tower is 17° 35′ and of the other tower 24° 48′. Find the height of each tower.

8. From a point on level ground 340.3 feet from the foot of a tower the angle of elevation of the top of the tower is 21° 16′. Find the height of the tower.

9. If a flag pole 15 feet high surmounts the tower of Exercise 8, find the angle of elevation of the top of the flag pole from the same point that is used in that exercise.

10. Two sides of a parallelogram are 55.23 feet and 41.88 feet long respectively and their included angle is 115° 37.2′. Find the altitude drawn to the longer side. Find the area of the parallelogram.

11. The hypotenuse of a right triangle is 500 feet long and one of its acute angles is 28° 32′. Show that the perpendicular from the vertex of the right angle to the hypotenuse is 209.83 feet.

12. If in Exercise 11 the hypotenuse is c and the angle is α, show that the perpendicular is $c \sin \alpha \cos \alpha$.

13. Calculate the perimeter and area of a regular decagon circumscribed about a circle whose radius is 124.5 inches.

14. The equatorial radius of the earth being taken as 3956 miles, find the radius and the circumference of the 40th parallel latitude. Find the radius of the arctic circle.

15. From a point in the same horizontal plane with the foot of a tower the angle of elevation of its top is 11° 29′. From a point 100 ft. nearer to the foot of the tower the angle is 13° 18′. Find the height of the tower.

Ans. 144.5 ft.

16. From one bank of a river the angle of elevation of the top of a tree on the opposite bank is $40° 22'$. On moving back 120 ft., the angle of elevation is $29° 37'$. Find the height of the tree and the width of the river.

17. At a certain point in the same horizontal plane with the foot of a column 25 ft. high, the angle of elevation of its top is $50°$. What will be the angle of elevation at a point 15 ft. farther away? *Ans.* $34° 48'$.

18. A column 75 ft. high stands on a pedestal 25 ft. high. From a certain point on the ground in the same horizontal plane with the foot of the pedestal, the latter subtends an angle of $15°$. What angle does the column subtend at this point? *Ans.* $31° 58.5'$.

19. A vertical pole 30 ft. long, and standing on level ground, casts a shadow 50 ft. long. What will be the length of the shadow when the sun is $10°$ higher?

20. From a point on the bank of a river the angle of elevation of the top of a tree on the opposite bank is $38° 52'$; from a point 200 ft. straight back from the bank the angle of elevation is $19° 26'$. Find the height of the tree and the width of the river. Also give graphic solution.

21. From a point A on level ground due south of an airplane, its angle of elevation is $41° 12'$; from a point B 1000 feet due east of A, the angle of elevation is $36° 41'$; how high is the airplane?

CHAPTER

V

PROJECTION OF LINE SEGMENTS. VECTORS. APPLICATIONS.

50. Projection of line segments.

Let PQ be a segment of a straight line and let AB be another straight line. The *projection* of segment PQ on line AB is the segment MN of line AB contained between the feet of the perpendiculars dropped from P and Q on AB. (Fig. 38.)

Along line HK we shall regard the direction *from P toward Q* as *positive.* Along line AB either direction may be chosen as positive. We choose it in the direction from A toward B. Positive directions may be conveniently indicated by arrows.

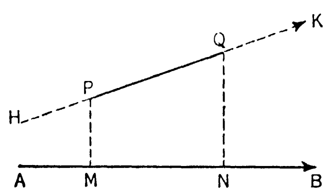


FIG. 38

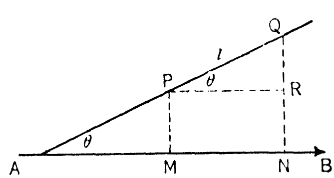


FIG. 39

The angle between segment PQ and line AB will be taken as the angle between their positive directions, measured counterclockwise from AB as initial line and with PQ (or PQ produced) as terminal line. Designate this angle by θ.

In Fig. 39 PR is drawn parallel to AB and θ = angle RPQ. In Fig. 40 PT is drawn parallel to AB and θ = angle TPQ, an obtuse angle. Line PT produced to the left to meet NQ determines PR.

Let l be a *positive* number which measures the length of segment PQ. Then we have the formula

$$(I) \qquad MN = l \cos \theta = \text{projection of } PQ \text{ on } AB.$$

Analysis of this formula.

In Fig. 39, from $\triangle\, RPQ$, $PR = l \cos \theta$. Angle θ is in the first quadrant, $\cos \theta$ is +, l is +, therefore PR comes out +. The arrow on PR points in the positive direction of AB. Also $MN = PR$ and is positive.

Let segment PQ rotate about P until $\theta = 90°$. Then R coincides with P and N with M. $PR = 0$ and $MN = 0$. The formula gives $MN = l \cos 90° = 0$.

When θ passes 90°, $\cos \theta$ becomes negative, as do PR and MN. (Fig. 40) When $\theta = 180°$, $MN = l \cos 180° = -l$. In the third quadrant $\cos \theta$, PR and MN remain negative; in the fourth quadrant all are positive.

Therefore our formula gives the projection of PQ on AB both as to length and sign.

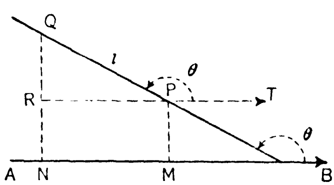


FIG. 40

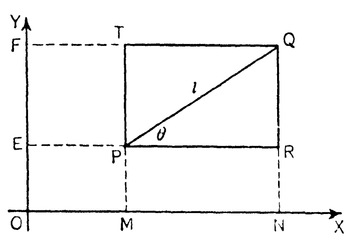


FIG. 41

In later work we shall need to project a given line segment on each of two mutually perpendicular lines.

Let these lines be OX and OY (Fig. 41) with positive directions as shown by the arrows. Let PQ be a given segment, making angle θ with OX.

Let MN be the projection of PQ on OX and EF its projection on OY. Then

(1) *projection of PQ on OX* $= MN = l \cos \theta$;

(2) *projection of PQ on OY* $= EF = l \sin \theta$.

Analysis of equation (2).

The figure shows θ to be an acute angle and $\triangle\ PRQ$ gives

$$l \cos \theta = PR = MN.$$
$$l \sin \theta = RQ = PT = EF.$$

These are equations (1) and (2) for angle θ acute. We have already shown that (1) remains true when θ varies from 0° to 360°. In exactly the same way we can show that equation (2) is true for all values of θ. This is left as an exercise for the student.

EXERCISES 21

Calculate the projections of PQ on OX and OY:

1. $PQ = 100$; $\theta = 0°, 30°, 45°, 60°, 90°, 150°, 240°, 270°, 300°$.
2. $PQ = 100$; $\theta = \frac{\pi}{5}, \frac{\pi}{9}, \frac{\pi}{7}, \frac{2\pi}{3}, \frac{9\pi}{7}, \frac{14\pi}{7}$.
3. $PQ = 356.2$; $\theta = 40° 15'$; $\theta = 205° 23'$, $\theta = -40° 15'$.
4. $PQ = 0.036825$; $\theta = 130° 45.3'$; $\theta = -130° 45.3'$.

51. Vectors and their components.

DEFINITIONS.

A ***vector*** *is a directed line segment.*

In Fig. 42, PQ is a vector, P is its *initial point* and Q is its *terminal point* or *end point.*

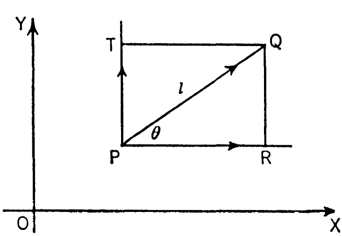


FIG. 42

Through initial point P draw a line in any desired direction and project PQ on that line. This projection of PQ is the *component of vector PQ in the desired direction.*

If two mutually perpendicular reference lines OX and OY be chosen, and lines parallel to them be drawn through P, the projections of PQ on these lines are PR and PT respectively. Then

PR is the *x-component* of vector PQ;
PT is the *y-component* of vector PQ.

Then for all values of θ we have (§50)

(1). *x-component of vector PQ* $= PR = l\cos\theta$;
(2). *y-component of vector PQ* $= PT = l\sin\theta$.

Also:

(3). *length of vector:* $l^2 = PR^2 + PT^2 =$ *sum of squares of components.*

(4). *angle of vector:* $\tan\theta = \dfrac{RQ}{PR} = \dfrac{PT}{PR} = \dfrac{y\text{-component}}{x\text{-component}}$.

NOTE. Point O may be taken at P, the initial point of the vector PQ; then OX falls on PR and OY on PT. This is done in the following section.

52. Sum of vectors. Parallelogram law. Resultant.

Problem. An airplane flies 125 miles in the direction E 34° N, then 150 miles in the direction E 62° N. How far and in what direction is the plane from its starting point?

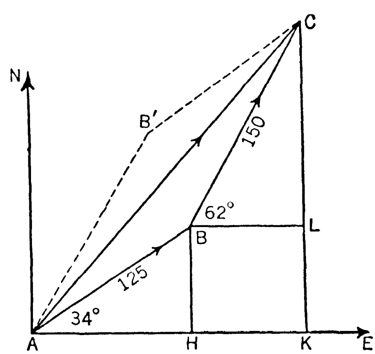


FIG. 43

The data are shown in Fig. 43. Vector AB has $l_1 = 125$, $\theta_1 = 34°$; vector BC has $l_2 = 150$, $\theta_2 = 62°$.

Required: l and θ for vector AC.

Calculate:

AH = x-comp. of vector AB $= l_1 \cos\theta_1$ $= 103.6$ mi.
BL = x-comp. of vector BC $= l_2 \cos\theta_2$ $= 70.4$ mi.
AK = x-comp. of vector AC $= AH + HK$ $= 174.0$ mi.

Similarly:

$KC = KL + LC = HB + LC = 69.9 + 132.4 = 202.3$ mi.

Knowing the components of vector AC we find l and θ by (3) and (4) of §51.

Exercise. Show that $l = 266.9$ miles; $\theta = 49° 18'$.

Sum of two vectors. In Fig. 43 *vector AC is called the sum of vectors AB and BC.* As an equation we write *vector AC = vector AB + vector BC.* When two vectors are added the final point of the first vector is taken as the initial point of the second vector.

Resultant of two vectors. In Fig. 43 AB' is drawn parallel to and equal to vector BC. Then the components of AB', regarded as a vector with initial point A, will be equal in length and direction to the components of vector BC. Therefore the components of vector AC may be obtained by adding the corresponding components of vectors AB and AB'. *Vector AC is called the resultant of vectors AB and AB'.*

Vector AC is the sum of vectors AB and BC, which are placed end to end; it is the resultant of vectors AB and AB' which start from the same initial point.

Parallelogram law. Vector AC is the diagonal of a parallelogram constructed on AB and BC, or on AB and AB', as sides. This is known as the *parallelogram law.*

In the following exercises a vector is indicated by the symbol (l, θ), where l is the length of the vector and θ is the angle which it makes with a selected initial line.

EXERCISES

Find the sum of each pair of vectors. Draw figures to scale.

1. (125, 34°) and (50, 62°).

2. (125, 34°) and (150, 120°).

3. (100, 60°) and (50, 150°).

4. (25, 145°) and (40, 210°).

5. (40, 240°) and (60, 120°).

6. (40, 240°) and (60, 30°).

7. (75, 300°) and (80, 225°).

8. (225, – 60°) and (125, 90°).

9. In these exercises would the answer be changed by reversing the order of the vectors?

10. How would the resultant of any pair of these vectors compare with their sum?

53. Velocities as vectors.

Suppose a ship to be moving at the rate of 20 knots an hour in the direction E 40° N. See Fig. 44. Let A mark its position at any moment and draw the directed line segment AB with $l = 20$ and $\theta = 40°$, choosing a convenient scale for l. Then AB is a vector showing both the speed and the direction of motion of the ship.

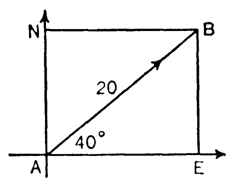


FIG. 44

NOTE. *Velocity* is commonly used to include both *speed* or *rate* of motion, and *direction* of motion.

As a second example consider an airplane flying at 200 miles an hour in a direction S 60° W. The vector diagram is shown in Fig. 45.

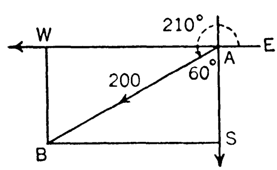


FIG. 45

In both figures we shall consider east and north as the positive directions along the reference lines. Angle θ is to be counted from the easterly direction as initial line. In Fig. 44, $l = 20$, $\theta = 40°$.

The components of vector AB are:

$$\text{Fig. 44: } AE = 20 \cos 40° = 15.32 \text{ knots per hour,}$$
$$AN = 20 \sin 40° = 12.86 \text{ knots per hour.}$$

$$\text{Fig. 45: } AW = 200 \cos 210° = -173.2 \text{ m.p.h.}$$
$$AS = 200 \sin 210° = -100.0 \text{ m.p.h.}$$

Note that, when the initial line points east:

> components to the east or north are counted positive;
> components to the west or south are counted negative.

Resultant of two velocities. Ground speed of airplane.

Problem. An airplane is traveling with an airspeed of 120 m.p.h. and heading E 50° N and the wind is blowing at 40 m.p.h. in direction N 20° W. Calculate the groundspeed and its direction.

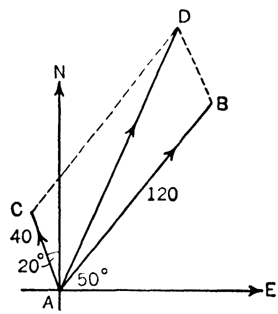


FIG. 46

(Airspeed = speed relative to the air. Groundspeed = speed relative to the ground.)

Starting from A, in one hour the engines would drive the plane from A to B while the wind would carry the plane from A to C. The plane follows the intermediate path AD and in one hour arrives at D. Vector AD, the distance covered in one hour relative to the ground, is the groundspeed.

EXERCISES 22

1. As in §52, calculate l and θ for vector AD, taking eastward as $\theta = 0$.

Calculate the groundspeed and direction from the data below.

	Airspeed	*Heading*	*Wind*	*Direction*
2.	125 m.p.h.	N 60° E	24 m.p.h.	N 45° W
3.	200	N 25° E	20	N 60° E
4.	180	S 50° E	30	N 65° W
5.	150	N 40° E	18	N 30° W
6.	140	S 55° W	22	N 70° W
7.	220	S 35° E	25	N 40° W

54. Forces as vectors.

Suppose a particle at A to be pulled upon by several forces, all in the same plane, as AF_1, AF_2, AF_3, AF_4 in the figure. Here each force is represented by a vector, showing the amount and direction of the pull.

What must be the amount and direction of a single force

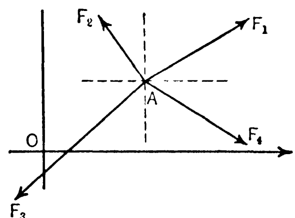


FIG. 47

which is equivalent to the four given forces? This is called the *resultant* of the given system of forces.

DEFINITION. The *sum* or *resultant* of any number of co-planar forces is a force such that

its x-component = sum of x-components of the given forces;
its y-component = sum of y-components of the given forces.

Solution. Resolve each force into an x-component and a y-component. This is done by the formulas

$$x\text{-component} = |\text{force}| \times \cos \alpha;$$
$$y\text{-component} = |\text{force}| \times \sin \alpha;$$

here $|\text{force}|$ denotes the magnitude of the force or the length of the vector which represents the force, and α is the angle between OX and AF, measured in the counter-clockwise direction. Thus for AF_4, $\alpha = 330°$ nearly.

Form the sum of the x-components, each with its proper sign, for a "total x-component." Similarly for the y-components. Then

$$\text{Amount of Resultant Force} = \sqrt{(\text{total } x\text{-comp.})^2 + (\text{total } y\text{-comp.})^2};$$

$$\text{Angle of Resultant Force:} \quad \tan \alpha = \frac{\text{total } y\text{-component}}{\text{total } x\text{-component}}.$$

EXERCISES 23

Calculate the resultant of each of the following systems of co-planar forces acting at a point. Draw accurate figures.

1. (30 lb., 25°); (40 lb., 50°).
2. (25 lb., 40°); (18 lb., 70°); (35 lb., 160°).
3. (75 lb., 65°); (60 lb., 130°); (85 lb., 230°); (40 lb., 340°).
4. Show that the resultant of two forces is represented by the diagonal of a parallelogram whose sides represent the two forces. (The parallelogram law.)

55. Plane surveying.

This subject furnishes further applications of the use of vectors and their components.

Suppose a surveyor to start from A and run the following lines:

	Bearing	*Distance*
A to B,	N 70° E,	345 feet;
B to C,	N 25° W,	288 feet;
C to D,	S 72° W,	467 feet;
D to E,	S 12° W,	424 feet.

How far and in what direction is he now from his starting point?

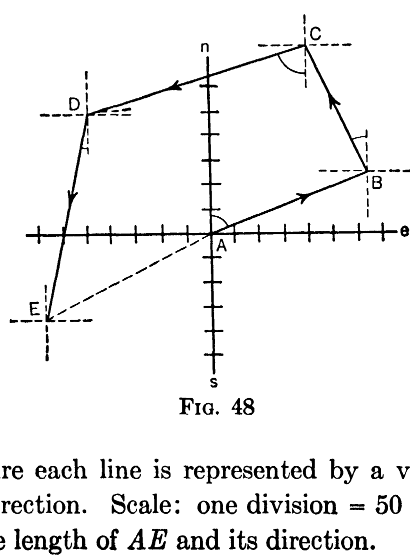


FIG. 48

In the figure each line is represented by a vector of proper length and direction. Scale: one division = 50 feet. We must determine the length of AE and its direction.

To do this we calculate the north-south component of each line and add them algebraically to get the north-south component of AE. These components are counted positive when the end point of the line lies to the north of its initial point; otherwise negative. The east-west components are treated similarly; they are counted positive when the end point of the line lies to the east of the initial point of the line.

DEFINITIONS. *The angles of the lines are measured from the north or the south, so that they will be acute angles. They are called the " bearings " of the lines. The distance run, or length of the vector, is called the " distance," and is assumed to lie in a horizontal plane.*

Also

the north-south component of a line is called its " *latitude* ";
the east-west component of a line is called its " *departure* ".

Then we have

latitude of a line $= \pm$ (distance $\times$ cosine of bearing);
departure of a line $= \pm$ (distance $\times$ sine of bearing).

Also

latitude of AE = algebraic sum of latitudes of lines run;
departure of AE = algebraic sum of departures of lines run.

$$\text{Distance } AE = \sqrt{(\text{latitude of } AE)^2 + (\text{departure of } AE)^2};$$

$$\text{Bearing of } AE\text{: } \tan sAE = \left| \frac{\text{latitude of } AE}{\text{departure of } AE} \right|,$$

where the vertical bars mean that the enclosed quantity is to be taken positively.

When a surveyor runs a closed traverse, starting at a given point and ending at the same point, the sum of the latitudes of all lines run should be zero, as also the sum of all the departures. This furnishes a check on the accuracy of the measurements. Exercises 2 and 3 below contain data from surveys of closed traverses.

Exercise 1. Calculate the latitude, departure, length and bearing of *AE* from the data given above. A figure drawn to scale may be measured to get approximate results.

Exercise 2. Check the computed latitudes and departures.

Line	Bearing	Distance (feet)	Latitude N	Latitude S	Departure E	Departure W
E–F	S 6° 44′ E	279.15		277.21	32.73	
F–V	N 54° 30′ W	153.27	89.00			124.78
V–U	S 16° 22′ W	120.17		115.29		91.25
U–M	N 23° 13′ W	231.47	212.73			33.86
M–E	N 67° 21′ E	235.42	90.66		217.26	
			+ 392.39	− 392.50	+ 249.99	− 249.89
				Error: − 0.11 ft.		Error: + 0.10 ft.

Exercise 3. Calculate the latitudes and departures.

Line	Bearing	Distance (feet)
A–B	S 86° 17′ W	267.23
B–C	N 14° 57′ W	228.15
C–D	N 0° 54′ E	261.72
D–E	S 89° 48′ E	134.53
E–F	S 2° 03′ E	230.43
F–G	S 85° 04′ E	174.46
G–A	S 1° 18′ E	219.07

56. Plane sailing.

The problem of plane sailing in navigation is essentially the same as the problem in plane surveying just treated. The surface of the ocean is considered as a plane.

DEFINITION. *The angle between the direction in which a ship is headed and the meridian passing through the ship's position is called the* **course** *of the ship. When measured from the nearer part of the meridian so as to be an acute angle, it corresponds exactly to* **bearing** *in surveying.* (See also §139).

Other elementary problems in navigation relate to the determination of the distance at which a ship, sailing a known course, will pass an observed object such as a lighthouse.

The term " bearing " which occurs in the exercises below means " *bearing off the bow,*" that is, the angle between the line from ship to object and the direction in which the ship is headed. When the bearing is 90° the object is "on the beam."

EXERCISES 24

1. A ship leaves Boston Light and sails S 75° E, 25 miles; then N 70° E, 40 miles; then N 35° E, 60 miles. In what direction should she now sail to return directly to the starting point? How far will she have to go?

2. A ship, sailing on track AB, is at A when the navigator observes the bearing of a lighthouse L to be 45° off the port bow; that is, the angle between the direction in which the ship is sailing and line AL is 45°. At what distance will the ship pass the lighthouse?
Ans. The distance sailed while the bearing increases from 45° to 90°.

3. A ship running on line AB is at A when the navigator observes the bearing of a lighthouse L to be 26.5° off the port bow. After a run of 5 miles the bearing has increased to 45°. Show that the distance at which the lighthouse will be passed is 5 miles very nearly. In general, if $AB = m$, also $BC = m$ and $CL = m$, *very nearly*.

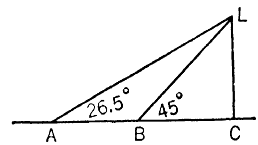


Fig. 49a

4. In Exercise 3 if $AB = BC = CL$, then $\tan BAL = \frac{1}{2}$. Why? How close is this to 26.5°?

5. A ship is running on line AB at 18 knots per hour. At A the navigator measures the bearing of lighthouse L to be 25°. Ten minutes later, at B, the bearing is 50°. How far is the ship from L at the time of the second bearing? At what distance will the ship pass the lighthouse?

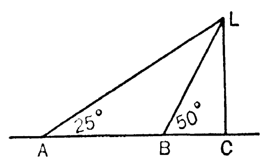


Fig. 49b

6. Solve Exercise 5 when angle $CAL = \alpha$, angle $CBL = 2\alpha$, and distance sailed $AB = m$. This method of finding CL is known as the "double angle method."

7. If, in the figure of Exercise 5, angle $CAL = 30°$, and, 20 minutes later, angle $CBL = 50°$, find CL.

8. If, in the figure of Exercise 5, angle $CAL = \alpha$, angle $CBL = \beta$, v is the speed of the ship in knots per hour and t the running time from A to B in minutes, find CL.

9. If a landmark is observed $22\frac{1}{2}°$ off the bow, and later 45° off the bow, show that the mark will be passed at distance approximately equal to seven tenths of the run of the ship between bearings. How accurate is this "seven tenths rule"?

10. If the first bearing is 20°, what must be the second bearing so that the "distance passed" shall be one half the run of the ship between bearings? *Ans.* 53° nearly.

11. A lighthouse tower rises 150 feet above sea level. There is shallow water out to a distance of 5160 feet from the tower. A navigator from his bridge 30 feet above sea level observes the angle of elevation of the light to be 1° 10′. How far out from the shoal is his ship?

57. Simple waves.

The graph of the function

$$y = \sin x$$

is a wave curve of the simplest type, as shown in the figure on p. 19.

Such a curve may be altered in several ways without destroying its simple wave form. We may change

(a) the height of the crests, or **amplitude** of the wave;
(b) the **length** of the wave;
(c) the **phase** of the wave, depending on where it cuts the x-axis.

In this way we would get a wave like that in the following figure, where the original sine wave is shown for comparison.

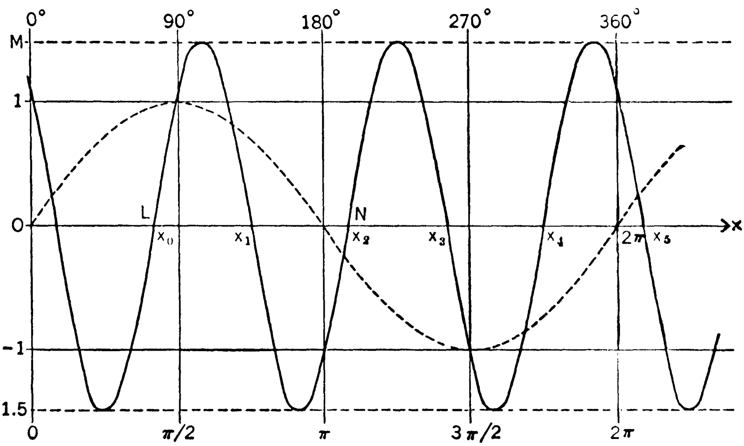


Fig. 50

Full line, $y = 1.5 \sin (3x - 4)$. *Dotted line,* $y = \sin x$.

Amplitude $= OM = 1.5$. Amplitude $= 1$.

Wave length $= LN = \frac{2\pi}{3}$ radians. Wave length $= 2\pi$ radians.

Phase $= OL = \frac{4}{3}$ radians. Phase $= 0$.

The most general expression for the simple wave which results when the above changes have been made in the wave for $\sin x$ is

$$y = k \sin (ax + b).$$

Example.

$$y = 1.5 \sin (3x - 4). \quad \text{(See preceding figure.)}$$

(a) *The amplitude.* The greatest value of sin $(3x - 4)$ is 1, since the sine function cannot exceed this value; hence the greatest value of y is 1.5. This shows the height of the wave; that is,

$$\text{amplitude} = 1.5.$$

(b) *The wave-length.* This is determined by finding the points where the wave crosses the x-axis. These are marked by the values of x for which $\sin (3x - 4) = 0$. But this is zero when the angle $(3x - 4)$ is an integral multiple of π;

$$\sin (3x - 4) = 0 \quad \text{if} \quad 3x - 4 = m\pi, \text{ or } x = \frac{m\pi + 4}{3} \text{ radians,}$$

where m is any whole number.

Putting $m = 0, 1, 2, 3, \ldots$, we get the successive crossing points:

$$x_0 = \frac{4}{3};\ x_1 = \frac{\pi + 4}{3};\ x_2 = \frac{2\pi + 4}{3};\ x_3 = \frac{3\pi + 4}{3};\ \text{etc.}$$

These values in degrees are, very nearly,

$$x_0 = 76.4°;\ x_1 = 136.4°;\ x_2 = 196.4°;\ x_3 = 256.4°;\ \text{etc.}$$

The distance between alternate crossing points, as x_0 to x_2, is the wave-length:

$$\text{wave-length} = x_2 - x_0 = \frac{2\pi + 4}{3} - \frac{4}{3} = \frac{2\pi}{3} \text{ radians.}$$

This is one-third of the wave-length of the fundamental sine wave.

(c) *The phase.* This marks the beginning of the first complete wave. Hence

$$\text{phase} = x_0 = OL = \tfrac{4}{3} \text{ radians.}$$

In general, for the wave $\quad y = k \sin (ax + b)$,

$$\text{amplitude} = k;$$

$$\text{wave-length} = \frac{2\pi}{a} \text{ radians;}$$

$$\text{phase} = -\frac{b}{a} \text{ radians.}$$

EXERCISES 25

Draw the following waves, showing each in comparison with $y = \sin x$.

1. $y = 2 \sin (x - 1)$.

2. $y = 3 \sin (2x - 3)$.

3. $y = 2.5 \sin (2x + 3)$.

4. $y = 4 \sin (3x - 60°)$.

5. Prove the statements made above regarding $y = k \sin (ax + b)$.

6. Draw the graph of $e = 110 \sin (240\pi t - \pi)$.

This equation describes the rise and fall of the electromotive force at a fixed point in an ideal alternating current circuit.

Here e and t take the place of y and x; e stands for electromotive force in volts, t for the time in seconds. Show that the greatest value of the electromotive force is 110 volts, and that there will be 120 vibrations per second.

In drawing the graph, use care in the choice of scales. Thus on cross-section paper, one square of the vertical scale might be taken to represent 10 volts, and 10 squares of the horizontal scale might be taken to represent $\frac{1}{120}$ seconds.

58. Simple harmonic motion.

Consider a point M to move on the circumference of a circle of radius r; we see that, as M moves around the circle, its projection M' moves back and forth along AC.

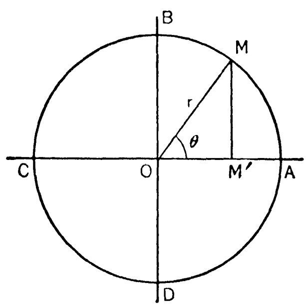


FIG. 51a

When $\theta = 0$, M and M' are together at A. Then, if we suppose θ to increase uniformly, M will move around the circle with uniform speed; but M' will move along AC with variable speed, slowly at first, then faster until it reaches O, when its speed will be greatest, then more slowly until it reaches C, where it will come to rest and start back toward A. This type of motion is called *simple harmonic motion.*

A body which has this motion vibrates back and forth past a middle position with variable speed. The distance of the body from its mid-position is called its *displacement.* From the figure,

$$\text{displacement of } M' = d = OM' = r \cos \theta.$$

When $\theta = 0$, d reaches its greatest value r, which is called the *amplitude* of the vibration.

If we measure θ from some other fixed radius OA' in place of OA, we shall have

$$d = r \cos (\theta + \alpha).$$

The greatest value of d is now reached when $\theta + \alpha = 0$, or when $\theta = -\alpha$; this angle is called the *phase* of the vibration.

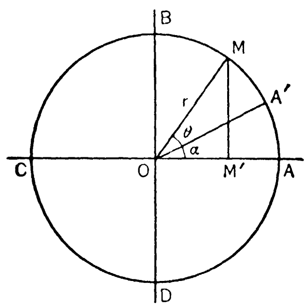


FIG. 51b

If we suppose the angular speed of the radius OM to be ω radians per second, and t to represent the time in seconds elapsed since M was at A, then $\theta = \omega t$, and

$$d = r \cos (\omega t + \alpha).$$

This is an equation of the type

$$y = k \cos (ax + b),$$

and, like the equation

$$y = k \sin (ax + b),$$

is represented graphically by a simple wave curve.

EXERCISES 26

Describe the motion of point M' when d is as follows.

1. $d = 2 \cos t$.

2. $d = 3 \cos 2t$.

3. $d = 5 \cos \pi t$.

4. $d = 5 \cos 2\pi t$.

5. $d = 10 \cos (\pi t + 45°)$.

6. $d = 10 \cos (2\pi t + 45°)$.

CHAPTER

VI

SMALL ANGLES. THE MIL UNIT. APPLICATIONS.

59. Use of small angles.

Consider a chord PQ of a circle of radius r. Let the length of the chord be small as compared with the radius of the circle. Then the central angle, θ, subtended by the chord will be small also.

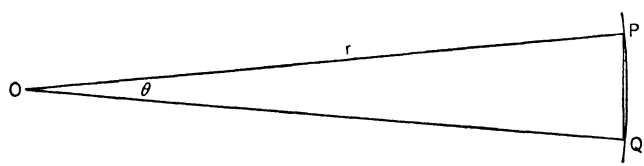


FIG. 52

We shall study the explanation of such a result as is given in the following problem.

Problem.

QP is a distant ship known to be 200 feet long. It subtends an angle of $1°$ as seen from O. How far away is the ship?

Ans. $57.3 \times 200 = 11460$ ft., very nearly.

Explanation. In §29(a) we have the relation between radius, arc, and radian measure of the central angle:

$$\text{(1)} \qquad \text{arc } QP = r \times \theta, \quad \text{or} \quad r = \frac{1}{\theta} \times \text{arc } QP.$$

If angle θ is small, arc QP will be nearly equal to chord PQ. Replacing arc QP by chord QP we have the *approximate* relations

$$\text{(2)} \qquad \text{chord } QP = r \times \theta, \quad \text{or} \quad r = \frac{1}{\theta} \times \text{chord } QP, \text{ approx.}$$

Taking $\theta = 1° = \frac{1}{57.3}$ radians, and chord $QP = 200$ feet, we have

$$r = 57.3 \times 200 = 11460 \text{ feet, approximately.}$$

In the next section we shall see how to obtain an estimate of the accuracy of this approximation. We shall find there that, if we use chord PQ in place of arc PQ,

the error in r is about 25 feet per mile if $\theta = 10°$;
the error in r is about 6.5 feet per mile if $\theta = 5°$;
the error in r is about $\frac{1}{4}$ foot per mile if $\theta = 1°$.

The following special cases of (2) may be noted.

(3) $\quad \theta = 1° = \frac{1}{57.3}$ radians; $r = 57.3 \times$ chord QP, (approx.).

(4) $\quad \theta = n° = \frac{n}{57.3}$ radians; $r = \frac{57.3}{n} \times$ chord QP, (approx.).

(5) $\quad \theta = 1' = \frac{1}{3440}$ radians; $r = 3440 \times$ chord QP, (approx.).

(6) $\quad \theta = n' = \frac{n}{3440}$ radians; $r = \frac{3440}{n} \times$ chord QP, (approx.).

Example.

A flagpole 12 feet long subtends at O an angle of $2°\ 30'$, point O being on a perpendicular bisector of the pole. How far is the pole from O?

Using formula (4) with $\theta = 2°\ 30' = 2.5°$, we have

$$r = \frac{57.3}{2.5} \times 12 = 275.0 \text{ feet.}$$

Using formula (6) with $\theta = 2°\ 30' = 150'$, we have

$$r = \frac{3440}{150} \times 12 = 275.2 \text{ feet.}$$

A right triangle solution with 5-place logarithms gives $r = 275.04$ feet.

EXERCISES 27

1. A chimney 40 feet high subtends an angle of 3°. How far away is the chimney?

2. A building 300 feet long viewed from a point at right angles to its length subtends an angle of $1°\ 45'$. How far away is the building?

3. At what distance from the building in Ex. 2 would the subtended angle be 2°?

4. A lighthouse tower 40 feet high subtends at a ship an angle of $30'$. How far is the ship from the lighthouse?

5. A textbook on navigation states that a certain light, 167 feet above sea level, will subtend an angle of 19′ at a distance of 5 miles. Check this statement. (See Ex. 4, §31.)

6. How many minutes in the angle subtended by a target 1 yard in diameter when viewed from a distance of 1000 yards?

7. How many minutes in angle θ if $r = 1000\ QP$? (Fig. 52)

8. Show that a ball, viewed from a distance equal to 57 times its diameter, will subtend at the eye an angle of nearly 1°; at a distance of 3400 times the diameter the angle will be very nearly 1′; at a distance of 206,000 times its diameter the angle will be almost exactly 1″.

9. At what distance from the eye will a baseball subtend an angle of 1°? Of 1′? Of 1″? (Diameter of baseball = 2.9 in.)

10. The moon's diameter is 2160 miles, the sun's 866,000 miles. Their distances from the earth are 240,000 miles and 93,000,000 miles respectively. What is the angular diameter of each body as viewed from the earth?

11. Is the end of a lead pencil, held at arm's length, sufficient to cover the disk of the full moon? Moon's angular diameter is 32′.

60. The limit of the ratio $\frac{\sin \alpha}{\alpha}$.

In a circle of radius r (Fig. 53) let QP be a chord, QNP its arc, 2α its central angle, and ST a segment of the tangent line at N.

From geometry, the length of the arc QNP is greater than the chord QP and less than the tangent ST. Taking half of each of these lengths we have

$$MP < \text{arc } NP < NT.$$

Dividing by r:

$$\frac{MP}{r} < \frac{\text{arc } NP}{r} < \frac{NT}{r}.$$

But

$$\frac{MP}{r} = \sin \alpha;$$

$$\frac{\text{arc } NP}{r} = \alpha \text{ (radians)};$$

$$\frac{NT}{r} = \frac{NT}{ON} = \tan \alpha.$$

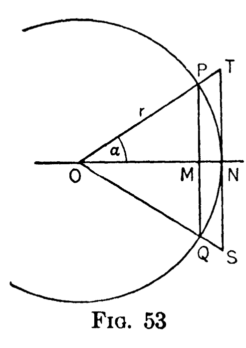


Fig. 53

Therefore, α being an acute angle measured in radians,

$$\sin \alpha < \alpha \text{ (radians)} < \tan \alpha.$$

Example.

$\alpha = 10°$; $\sin\alpha = 0.1736$, $\alpha = 0.1745$, $\tan\alpha = 0.1763$.

Dividing the preceding inequalities by $\sin\alpha$,

$$1 < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}.$$

As angle α approaches 0, $\frac{1}{\cos\alpha}$ approaches 1, and the intermediate quantity, $\frac{\alpha}{\sin\alpha}$, must likewise approach 1. Also the reciprocal quantity $\frac{\sin\alpha}{\alpha}$ must approach 1.

THEOREM. *The limit of the ratio* $\frac{\sin\alpha}{\alpha}$, *as* α *approaches* 0, *is* 1, α *being measured in radians.*

COROLLARY. *When angle* α *is quite small, the ratio* $\frac{\sin\alpha}{\alpha}$ *will differ only slightly from* 1.

We may therefore write

$$\frac{\sin\alpha}{\alpha} \doteq 1 - e, \text{ where } e \text{ is a small positive number,}$$

and

$$\sin\alpha = \alpha - e\alpha.$$

If we neglect the small quantity $e\alpha$, we have

$$\sin\alpha = \alpha \text{ (radians), approximately.}$$

If α *is a small angle,* $\sin\alpha$ *and radian measure of* α *are nearly equal.*

From tables we can take the values of α and $\sin\alpha$ and calculate e. We find, in round numbers,

$$e = \frac{1}{200} \text{ approx. when } \alpha = 10°,$$

$$e = \frac{1}{800} \text{ approx. when } \alpha = 5°,$$

$$e = \frac{1}{20{,}000} \text{ approx. when } \alpha = 1°.$$

Up to 5° the values of α (rad.) and $\sin \alpha$ are so nearly equal that we can use them interchangeably in many applications. It amounts to replacing a short chord of a circle by its arc or vice versa, because, in Fig. 53, $r \sin \alpha$ is the half-chord and $r\alpha$ the half-arc.

In the problem of §59 we had chord $QP = 200$ feet, $\theta = 1° = \frac{1}{57.3}$ radians. Now arc $QP = r\theta$. Replace arc QP by chord QP; then

$$200 = r \cdot \frac{1}{57.3};$$

$$r = 57.3 \times 200.$$

The error in r due to using chord QP in place of arc QP is about 1 part in 20,000 or $11{,}460 \div 20{,}000 = 0.6$ foot. There is of course also a slight error due to the use of 57.3 in place of of $\frac{180}{\pi}$.

The limit of the ratio $\frac{\tan \alpha}{\alpha}$.

If we divide the inequalities

$$\sin \alpha < \alpha < \tan \alpha$$

by $\tan \alpha$, we obtain

$$\cos \alpha < \frac{\alpha}{\tan \alpha} < 1.$$

From these inequalities we derive, by the reasoning used above, the following theorem.

THEOREM. *The limit of the ratio* $\frac{\tan \alpha}{\alpha}$, *as α approaches 0, is 1, α being measured in radians.*

COROLLARY. *When angle α is quite small, the ratio* $\frac{\tan \alpha}{\alpha}$ *will differ only slightly from 1.*

We may therefore write

$$\frac{\tan \alpha}{\alpha} = 1 + e, \text{ where } e \text{ is a small positive number;}$$

or,

$$\tan \alpha = \alpha + e\alpha.$$

If we neglect the small quantity $e\alpha$, we have

$$\tan \alpha = \alpha \text{ (radians), approximately.}$$

If α is a small angle, tan α *and the radian measure of α are nearly equal.*

The values of e for $\alpha = 10°, 5°, 1°$, respectively, are practically the same as those stated above for sin α.

Values of S and T. For small angles, less than 5°, the values of log sin α and of log tan α can not be obtained accurately by interpolation in the tables. To obtain more accurate values, the preceding approximations for sin α and tan α are used. We consider first the case of sin α, when angle α is small and is expressed in minutes.

If α represents the number of radians in our angle and α' the number of minutes, we have $\alpha = \frac{\pi}{10800}\alpha'$, and therefore

$$\sin \alpha = \alpha = \frac{\pi}{10800}\alpha', \text{ approximately.}$$

Therefore

$$\log \sin \alpha = \log \alpha' + \log\left(\frac{\pi}{10800}\right), \text{ approximately.}$$

If we write

$$\log \sin \alpha = \log \alpha' + S,$$

the value of S will differ only slightly from $\log\left(\frac{\pi}{10800}\right)$. It is tabulated in Table II of Appendix B. To find the value of log sin α, when α is a small angle, add S to log α'.

In the same way we obtain

$$\log \tan \alpha = \log \alpha' + T,$$

where the values of T are likewise tabulated.

If a small angle is given to seconds we would proceed as above, but start with the relation $\alpha = \frac{\pi}{64800}\alpha''$ and use the corresponding values of S and T.

61. The mil unit of angle.

(I) If the circumference of a circle be divided into 6400 equal arcs, each arc will be equal, very nearly, to one one-thousandth part of the radius. The length of one such arc is equal to

$$\frac{\text{circumference}}{6400} = \frac{2\pi r}{6400} = \frac{r}{1000}, \text{ very nearly.}$$

A more accurate value is $\frac{r}{1018.6}$, but for practical applications the divisor 1018.6 is rounded off to 1000. This introduces an error of about one part in 50, or 2%.

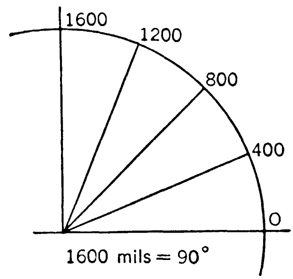


FIG. 54

The central angle subtended by an arc equal to one 6400th part of the circumference is called a ***mil***. It is the standard unit of angle in the artillery service.

We have then

$$6400 \text{ mils} = 360° = 2\pi \text{ radians.}$$
$$1600 \text{ mils} = 90° = \text{a quadrant.}$$
$$1 \text{ mil} = \frac{90 \times 60}{1600} \text{ minutes} = 3\tfrac{3}{8} \text{ minutes.}$$

For practical purposes we regard the mil as the central angle whose arc (or chord) is one 1000th part of the radius.

(II) *Applications involving small angles.*

According to the definition of the mil the following statements are approximately correct.

1) A target one yard in diameter and 1000 yards distant from a gun will subtend at the gun an angle of 1 mil, very nearly.

2) A target D yards in diameter and 1000 yards distant from a gun will subtend at the gun an angle of D mils, very nearly.

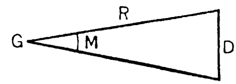
3) A target D yards in diameter and r yards distant from a gun will subtend at the gun an angle of $\frac{D}{r \div 1000}$ mils, very nearly.

The last two statements are quite accurate for angles up to 100 mils or 5.6°. Statement 3) may be written in the form

$$3') \qquad \text{mils subtended at gun} = \frac{\text{diameter of target}}{\text{range} \div 1000}.$$

If we let M represent the number of mils, D the diameter of target in yards, r the range in yards, and R the range in thousands of yards so that $R = r \div 1000$, we have

$$3'') \qquad M = \frac{D}{R}; \quad R = \frac{D}{M}; \quad D = MR.$$



Examples.

(a) How many mils will be subtended by a target 65 yards in diameter when the range is 2750 yards?

$$M = \frac{65}{2.750} = 24 \text{ mils.}$$

(b) What is the range when a target known to be 45 yards in diameter subtends an angle of 21 mils?

$$R = \tfrac{45}{21} = 2.143; \quad r = \text{range} = 2143 \text{ yards.}$$

EXERCISES 28

The first three exercises may be used for oral drill. Use should be made of the short cuts of arithmetic.

1. Determine M. **2.** Determine r. **3.** Determine D.

	D	r	D	M	r	M
1)	5	1000	15	5	2000	16
2)	25	10,000	15	6	4000	22
3)	20	8000	24	32	1800	15
4)	60	7500	40	12	1750	36
5)	40	6000	60	25	2250	60
6)	30	2500	90	36	3200	45
7)	15	1750	100	30	1625	54
8)	75	2450	72	27	2745	8
9)	125	3750	155	75	1275	64
10)	95	3800	120	32	2125	88

4. In Fig. 52, if the gun is at O and chord QP is the target,

$$\theta \text{ (mils)} = \frac{QP}{r \div 1000} \text{ approx.}; \quad \sin \tfrac{1}{2}\theta = \frac{\tfrac{1}{2}QP}{r} \text{ exactly.}$$

Take $QP = 180$ yards and $r = 2000$ yards; calculate θ from each of these equations and compare results.

5. State which of the following equations are exact and which are approximate; where approximate, give the exact value.

(a) $\frac{\pi}{4}$ radians $= 800$ mils;

(b) 160 mils $= 9°$;

(c) 1 degree $= 17.78$ mils;

(d) 1 radian $= 1000$ mils.

6. What is the distance to a ship which is known to be 300 feet long and which subtends an angle of 20 mils when viewed broadside on?

7. What angle in mils is subtended by a building 180 feet long when viewed broadside on from a distance of 1500 yards?

8. What angle is subtended at a target by a battery front of 80 yards, the target being 2400 yards distant in a direction perpendicular to the center of the battery front?

9. If a gun is sighted at a tree 2400 yards away and if a concealed target is known to be located 75 yards to the right of the tree, through what horizontal angle must the gun be deflected to bear in the direction of the target?

10. If the four guns of a battery are mounted at the vertices of a square 50 yards on a side and if a target is in line with one diagonal of the square and 2500 yards from its center, what angle is subtended at the target by the other diagonal of the square?

(III) In the artillery service the mil is used when angles are not small enough to permit the use of approximate methods. A brief table, Table *B*, to be used with the mil as argument, appears on page 89. The circular and radian values of the angles have been added merely for comparison.

EXERCISES 29

Determine angle M.

	(a)	(b)	(c)	(d)	(e)
1. $\sin M$	0.560;	0.130;	0.500;	0.930;	0.660.
2. $\cos M$	0.300;	0.500;	0.770;	0.912;	0.989.
3. $\tan M$	0.220;	0.100;	0.620;	1.500;	2.00 .

Solve the following right triangles. The notation is as in Fig. 33. The symbol $\overline{\text{m}}$ is used for mil.

4. $c = 1800$; $\alpha = 600\overline{\text{m}}$.
5. $a = 125$; $\alpha = 740\overline{\text{m}}$.
6. $b = 250$; $\alpha = 900\overline{\text{m}}$.
7. $a = 1200$; $\beta = 300\overline{\text{m}}$.
8. $b = 2250$; $\beta = 250\overline{\text{m}}$.
9. $c = 1550$; $\beta = 1200\overline{\text{m}}$.
10. $c = 300$; $a = 200$.
11. $b = 150$; $c = 175$.
12. $a = 125$; $b = 150$.

13. From a battery position the inclined range to an airplane is found to be 4000 yards and its angle of elevation $540\overline{\text{m}}$. How high is the airplane? What is its horizontal range?

14. If an aiming point is 1500 yards from a gun and an invisible target is known to be 600 yards to the right of the aiming point as seen from the gun, what is the angle at the gun between direction of aiming point and direction of target?

15. In Fig. 55 take $OT = 3600$ yards, $OG = 1600$ yards, and angle $TOG = 2000\overline{\text{m}}$. Calculate HG, OH, GT, and angle GTO.

TABLE B

Mils	Degrees	Radians	Sine	Cosine	Tangent
0	0° 0′	.000	.000	1.000	.000
40	2 15	.039	.039	0.999	.039
80	4 30	.079	.079	.997	.079
120	6 45	.118	.118	.993	.118
160	9 00	.157	.156	.988	.158
200	11 15	.196	.195	.981	.199
240	13 30	.236	.233	.972	.240
280	15 45	.275	.271	.962	.282
320	18 00	.314	.309	.951	.325
360	20 15	.353	.346	.938	.369
400	22 30	.393	.383	.924	.414
440	24 45	.432	.419	.908	.461
480	27 00	.471	.454	.891	.510
520	29 15	.511	.489	.873	.560
560	31 30	.550	.523	.853	.613
600	33 45	.589	.556	.831	.668
640	36 00	.628	.588	.809	.727
680	38 15	.668	.619	.785	.788
720	40 30	.707	.649	.760	.854
760	42 45	.746	.679	.734	.924
800	45 00	.785	.707	.707	1.000
840	47 15	.825	.734	.679	1.082
880	49 30	.864	.760	.649	1.171
920	51 45	.903	.785	.619	1.267
960	54 00	.942	.809	.588	1.376
1000	56 15	.982	.831	.556	1.497
1040	58 30	1.021	.853	.523	1.632
1080	60 45	1.060	.873	.489	1.786
1120	63 00	1.100	.891	.454	1.963
1160	65 15	1.139	.908	.419	2.169
1200	67 30	1.178	.924	.383	2.414
1240	69 45	1.217	.938	.346	2.711
1280	72 00	1.257	.951	.309	3.078
1320	74 15	1.296	.962	.271	3.546
1360	76 30	1.335	.972	.233	4.165
1400	78 45	1.374	.981	.195	5.027
1440	81 00	1.414	.988	.156	6.314
1480	83 15	1.453	.993	.118	8.449
1520	85 30	1.492	.997	.078	12.71
1560	87 45	1.532	.999	.039	25.45
1600	90 00	1.571	1.000	.000	

62. Azimuths. Azimuth difference.

The direction of a line in a horizontal plane may be indicated by giving the angle which the line makes with a line of known direction. This angle is called the azimuth of the line.

Let O, G, T denote, respectively, the position of an observer, a gun, and a target. Let TS, due southward from T, be used as the reference line for azimuths. The observer at O knows the lengths and directions (azimuths) of lines OT and OG. He wishes to obtain the azimuth and length of GT for transmission to the gunner.

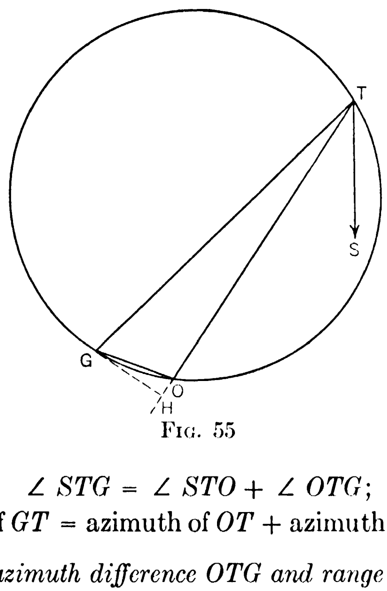


FIG. 55

Now $\angle STG = \angle STO + \angle OTG$;

or, azimuth of GT = azimuth of OT + azimuth difference OTG.

Calculation of azimuth difference OTG and range GT.

We assume that OT is large in comparison with OG, so that $\triangle OGT$ is a long slender triangle. Assume that the length of OT is less than the length of GT. Draw GH perpendicular to OT (produced), forming right triangle OGH, in which OG and $\angle GOH$ are known.

Then $HO = OG \cos GOH$ and $GH = OG \sin GOH$.

$$\text{Range } GT = HT \text{ approx.} = OT + OH = OT + OG \cos GOH.$$

$$\text{Azimuth diff. } OTG \text{ (mils)} = \frac{GH}{GT \div 1000} = \frac{OG \sin GOH}{GT \div 1000}.$$

Exercise. Calculate these quantities when azimuth of OT is 30°, OT = 2400 yards, OG = 300 yards, $\angle\ TOG$ = 100°. Repeat the calculations with the same data except that $\angle\ TOG$ is now 80°. The range correction HO will now be negative.

63. Parallax. Range finder.

If a target, T, (considered as a point) is viewed from two

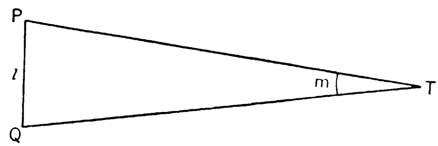


FIG. 56

points P and Q, angle PTQ is called the parallactic angle at T, or simply the parallax of T, due to line-segment PQ.

We shall assume that $PT = QT$, as in Fig. 56, where angle m is the parallax of T due to line-segment PQ of length l.

One type of range finder is an instrument which gives the range to a target by means of the angle subtended at the target by a tube of known length which forms part of the instrument. Two images of the target caught at the ends of the tube are brought to coincide by turning a milled head, the amount of turning depending on the parallactic angle, which in turn depends on the range.

Exercise 1. If l = 4 yards in Fig. 56, what range should correspond to each of the following parallaxes:

m = 1 mil; 5 mils; 15 mils; 3 mils; 7 mils?

Exercise 2. If l = 6 yards what are the parallaxes corresponding to the following ranges:

r = 1000 yards; 3000 yards; 1500 yards; 1200 yards; 2400 yards?

Parallax as used in Astronomy.

When the direction of the center of the sun, moon, or one of the nearer planets is observed from the surface of the earth a correction must be made to obtain the direction as it would be measured from the center of the earth. · This is due to the fact that astronomical tables give the position of the bodies of the solar system treating each body as a point.

Let O be a position of an observer on the earth's surface, OH a horizontal line, M the center of the moon, angle HOM the angle of elevation (altitude) of the moon's center above the horizon.

Then the difference of direction of M as seen from O and C is angle OMC, called the *parallactic angle* or merely the *parallax* of the moon at altitude θ. This is angle p in the figure.

When the center of the moon is on the horizon, the parallactic

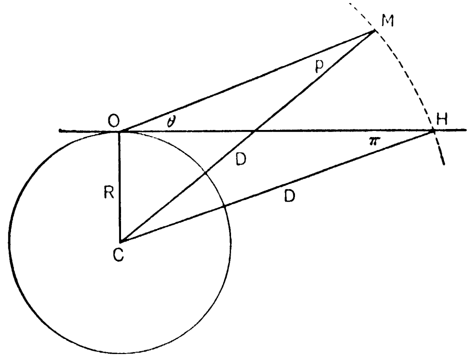


FIG. 56a

angle is OHC. This is called the moon's *horizontal parallax* and is represented by the letter π.

Take $CO = R = 4000$ miles and $CM = CH = 240{,}000$ miles. Then

$$\sin \pi = \pi \text{ (radians) approx.} = \tfrac{4000}{240000} = \tfrac{1}{60}$$

That is:

$$\begin{aligned} \text{angle } \pi &= \tfrac{1}{60} \text{ of a radian} \\ &= \frac{57.3}{60} \text{ degrees} = 0.95^\circ \\ &= \tfrac{3440}{60} \text{ minutes} = 57.3'. \end{aligned}$$

Exercise 1. The moon's distance from the earth varies from 221000 miles to 253000 miles. What is the corresponding range of variation of the moon's horizontal parallax expressed in minutes?

Exercise 2. For the sun, the distance D is 93,000,000 miles. Show that the sun's horizontal parallax is 8.8″, if $R = 3960$ miles.

Hint: Angle π in seconds $= 206000 \dfrac{R}{D}$.

Exercise 3. Calculate π for Mars, when at the distance $D = 50{,}000{,}000$ miles.

64. Path of projectile. Parabolic trajectory.

If a projectile leaves a gun at an angle θ with the horizontal and with a muzzle velocity v_0 feet per second (initial velocity or speed), the horizontal component of v_0 is $v_0 \cos \theta$. This represents the rate at which the projectile will progress in a horizontal direction and in t seconds the horizontal displacement of the projectile will be $tv_0 \cos \theta$ feet. (Air resistance neglected.)

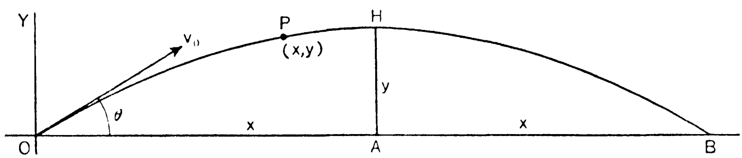


FIG. 57

The initial vertical speed will be $v_0 \sin \theta$ which, if unchecked, would give the height of the projectile in t seconds as $tv_0 \sin \theta$. But during t seconds gravity would cause the projectile to fall $\frac{1}{2}gt^2$ feet, ($g = 32.2$), so the net height is ($tv_0 \sin \theta - \frac{1}{2}gt^2$) feet. Hence in t seconds the "coordinates" of the projectile in feet will be (air resistance neglected),

$$x = tv_0 \cos \theta; \quad y = tv_0 \sin \theta - \tfrac{1}{2}gt^2.$$

Exercise. (a) Take $v_0 = 1200$ feet per second, and $\theta = 30°$. Calculate the values of the coordinates x and y for $t = 0, 5, 10, 15, 20, 25, 30, 35$ seconds. Plot the points (x, y).

(b) The curve so obtained is a parabola. The highest point will be reached in about 19 seconds. The exact value of the time of arrival at the highest point, call it T, will be $T = v_0 \sin \theta \div g$ because the initial vertical speed $v_0 \sin \theta$ is reduced at the rate of g feet per second. Calculate T with v_0 and θ as in (a).

(c) Having found T, we can find X and Y, the coordinates of the highest point of the trajectory:

$$X = Tv_0 \cos \theta; \quad Y = Tv_0 \sin \theta - \tfrac{1}{2}\, gT^2 = \frac{(v_0 \sin \theta)^2}{2g}.$$

Calculate X and Y. (The answers will be in feet.)

(d) The descending part of the trajectory (parabola) is symmetrical with the ascending part. Hence

$$\text{time of flight} = 2T = 2v_0 \sin \theta \div g;$$
$$\text{horizontal range } OB = 2X = 2Tv_0 \cos \theta.$$

Calculate the time of flight and the range.

CHAPTER

VII

FUNCTIONS OF SEVERAL ANGLES

65. Formulas for sin (x + y) and cos (x + y).

Let x and y be two angles, each of which we first assume to be less than 90°. Their sum will then fall in the first or the second quadrant. The two cases are illustrated in the figures, and the demonstration which follows applies to either figure.

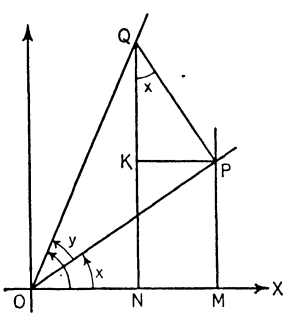


FIG. 58a

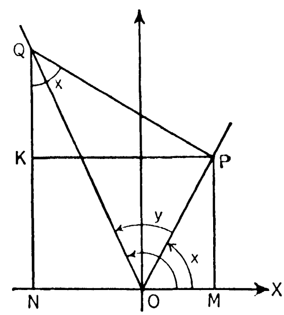


FIG. 58b

Construct $\angle\ XOP = x$ and $\angle\ POQ = y$, the terminal side of x being taken as the initial side of y.

From Q, any point on the terminal side of y, draw perpendiculars NQ and PQ to the sides of angle x, produced if necessary. Draw $MP \perp OX$ and $KP \perp NQ$.

Then $\angle\ KQP = x$, and in either figure,

$$\sin(x+y) = \frac{NQ}{OQ} = \frac{MP+KQ}{OQ} = \frac{MP}{OQ} + \frac{KQ}{OQ}$$

$$= \frac{MP}{OP}\cdot\frac{OP}{OQ} + \frac{KQ}{PQ}\cdot\frac{PQ}{OQ}.$$

Hence

(a) $\quad\quad$ ***sin (x + y) = sin x cos y + cos x sin y.***

Also, noting that ON in the second figure is a negative segment,

$$\cos(x+y) = \frac{ON}{OQ} = \frac{OM - NM}{OQ} = \frac{OM}{OQ} - \frac{KP}{OQ}$$
$$= \frac{OM}{OP} \cdot \frac{OP}{OQ} - \frac{KP}{PQ} \cdot \frac{PQ}{OQ}.$$

Hence

(b) $\qquad \boldsymbol{\cos(x+y) = \cos x \cos y - \sin x \sin y.}$

66. Generalization of formulas (a) and (b).

In the preceding proofs we assumed angles x and y to be acute angles. Geometric proofs may be made to show that formulas (a) and (b) hold for any two angles. We shall not do this, but instead, shall use the method of proof by induction.

We begin by showing that, if formulas (a) and (b) are true for two angles α and β, they will remain true when either angle is increased (or diminished) by 90°.

First we note two relations obtained by use of Rule (b) of §21. If θ is any angle,

(1) $\sin(\theta + 90°) = \cos\theta$; (2) $\cos(\theta + 90°) = -\sin\theta$.

Now we assume that the following equations hold for two angles α and β,

(a′) $\qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta;$

(b′) $\qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

We shall show that these equations remain true when angle α is increased by 90°. Accordingly we replace α by $\alpha' = \alpha + 90°$. We obtain

(a″) $\qquad \sin(\alpha'+\beta) = \sin\alpha'\cos\beta + \cos\alpha'\sin\beta;$

(b″) $\qquad \cos(\alpha'+\beta) = \cos\alpha'\cos\beta - \sin\alpha'\sin\beta.$

We wish to prove that the last two equations are true if the first two are true. Consider equation (a″).

The left hand side may be written, by equation (1),

$$\sin(\alpha'+\beta) = \sin(\alpha+\beta+90°) = \cos(\alpha+\beta).$$

The right hand side, by use of (1) and (2), becomes

$$\sin(\alpha+90°)\cos\beta + \cos(\alpha+90°)\sin\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Substituting these in (a″) we obtain (b′) which was assumed to be true. Hence (a″) is true.

In the same way we can show that (b″) is true.

We have now shown that if equations (a′) and (b′) are true in any given case, they remain true when either angle is increased by 90°.

But they are true when $\alpha = x$ and $\beta = y$, x and y being acute angles; hence they are true when $\alpha = x + 90°$; if true for $\alpha = x + 90°$, they are true for $x + 2 \cdot 90°$; and so on. Similarly for angle β.

In a similar manner it can be shown that (a′) and (b′) remain true when either angle is diminished by 90°.

Since any angle can be represented by $x \pm n \cdot 90°$ where x is an acute angle, we have proved that formulas (a′) and (b′) are true for all values of α and β.

Examples.

1. $\sin 75° = \sin (45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

2. $\cos 255° = \cos (225° + 30°) = \cos 225° \cos 30° - \sin 225° \sin 30°$

$$= -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{1}{2} = \frac{-\sqrt{6} + \sqrt{2}}{4}.$$

3. Given $\sin \alpha = \frac{3}{5}$, $\cos \beta = \frac{2}{3}$, both angles in the first quadrant. Calculate $\sin (\alpha + \beta)$.

From $\sin \alpha = \frac{3}{5}$ we find $\cos \alpha = \frac{4}{5}$;

from $\cos \beta = \frac{2}{3}$ we find $\sin \beta = \frac{\sqrt{5}}{3}$.

Substituting these in the formula (a), $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, we obtain

$$\sin (\alpha + \beta) = \frac{3}{5} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{\sqrt{5}}{3} = \frac{6 + 4\sqrt{5}}{15}.$$

4. Given $\sin \alpha = \frac{3}{5}$, α in the first quadrant; $\cos \beta = -\frac{2}{3}$, β in the third quadrant; calculate the value of $\cos (\alpha + \beta)$.

(b) $\qquad \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$

From the given data we find $\cos \alpha = \frac{4}{5}$, $\sin \beta = -\frac{\sqrt{5}}{3}$. Then

$$\cos(\alpha+\beta) = \frac{4}{5}\left(-\frac{2}{3}\right) - \frac{3}{5}\left(-\frac{\sqrt{5}}{3}\right) = \frac{-8+3\sqrt{5}}{15}.$$

5. Show that $\dfrac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta} = \cos\beta - \tan\alpha.$

$$\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\sin\beta}$$

$$= \frac{\cos\alpha\cos\beta}{\cos\alpha\sin\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\sin\beta}.$$

$$= \cot\beta - \tan\alpha.$$

6. Show that $\dfrac{\cos(45^\circ+A)}{\sin(45^\circ+A)} = \dfrac{1-\tan A}{1+\tan A}.$

$$\frac{\cos(45^\circ+A)}{\sin(45^\circ+A)} = \frac{\cos 45^\circ\cos A - \sin 45^\circ\sin A}{\sin 45^\circ\cos A + \cos 45^\circ\sin A}$$

$$= \frac{\dfrac{1}{\sqrt{2}}\cos A - \dfrac{1}{\sqrt{2}}\sin A}{\dfrac{1}{\sqrt{2}}\cos A + \dfrac{1}{\sqrt{2}}\sin A}$$

$$= \frac{\cos A - \sin A}{\cos A + \sin A} = \frac{1 - \dfrac{\sin A}{\cos A}}{1 + \dfrac{\sin A}{\cos A}}$$

$$= \frac{1-\tan A}{1+\tan A}.$$

NOTE. Formulas (a′) and (b′) should be learned in *verbal* form rather than in terms of any particular letters.

(a′) *The sine of the sum of two angles equals the sine of the first angle times the cosine of the second plus the cosine of the first angle times the sine of the second.*

(b′) Let the student give the verbal statement.

EXERCISES 30

1. $\sin 90^\circ = \sin(30^\circ + 60^\circ) = 1.$

2. $\cos 90^\circ = \cos(30^\circ + 60^\circ) = 0.$

3. $\sin 105^\circ = \sin(45^\circ + 60^\circ) = \frac{1}{4}(\sqrt{2} + \sqrt{6}).$

4. $\cos 105^\circ = \cos(45^\circ + 60^\circ) = \frac{1}{4}(\sqrt{2} - \sqrt{6}).$

5. $\sin 165^\circ = \sin(30^\circ + 135^\circ) = \frac{1}{4}(\sqrt{6} - \sqrt{2}).$

6. $\cos 165^\circ = \cos(30^\circ + 135^\circ) = -\frac{1}{4}(\sqrt{6} + \sqrt{2}).$

7. $\sin 285^\circ = \sin(60^\circ + 225^\circ) = -\frac{1}{4}(\sqrt{6} + \sqrt{2}).$

8. $\cos 285^\circ = \cos(60^\circ + 225^\circ) = \frac{1}{4}(\sqrt{6} - \sqrt{2}).$

Prove the identities of Exercises 9–17, using formulas (a) or (b).

9. $\sin(x + 90°) = \cos x$.

10. $\cos(x + 90°) = -\sin x$.

11. $\sin(x + 180°) = -\sin x$.

12. $\cos(x + 180°) = -\cos x$.

13. $\sin(x + 270°) = -\cos x$.

14. $\cos(x + 270°) = \sin x$.

15. $\sin(x + 30°) = \frac{1}{2}(\sqrt{3}\sin x + \cos x)$.

16. $\sin(x + 45°) = \dfrac{1}{\sqrt{2}}(\sin x + \cos x)$.

17. $\cos(x + 45°) = \dfrac{1}{\sqrt{2}}(\cos x - \sin x)$.

18. $\sin 2x = 2\sin x \cos x \quad [2x = x + x]$.

19. If $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{2}{3}$, α and β in quadrant I, calculate $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

20. If $\sin\alpha = \frac{4}{5}$, α in quadrant II, $\cos\beta = \frac{2}{3}$, β in quadrant IV, calculate $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

67. Formulas for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$.

Replacing β by $-\beta$ in (a) and (b), we have the two equations

$$\sin(\alpha - \beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta);$$
$$\cos(\alpha - \beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta).$$

But $\quad \cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$.

Therefore the two preceding equations become

(c) $\quad \boldsymbol{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta;}$

(d) $\quad \boldsymbol{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.}$

There are really two steps involved here:

1) in (a) and (b) replace β by $-\theta$ and reduce as above;

2) then replace the letter θ by the letter β, to conform to the letters used in (a) and (b).

Equations (a), (b), (c), (d) are usually called the addition and subtraction formulas of trigonometry. All the other working formulas are deduced from them.

Examples.

1. $\cos 75° = \cos(135° - 60°) = \cos 135°\cos 60° + \sin 135°\sin 60°$

$$= -\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

2. Given $\sin\alpha = \frac{3}{5}$, $\cos\beta = -\frac{2}{3}$; calculate all the values of $\cos(\alpha - \beta)$.

Angle α may lie in quadrant I or II; $\cos\alpha = \pm\frac{4}{5}$. Angle β may lie in quadrant II or III; $\sin\beta = \pm\dfrac{\sqrt{5}}{3}$.

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \pm\frac{4}{5}\left(-\frac{2}{3}\right) + \frac{3}{5}\left(\pm\frac{\sqrt{5}}{3}\right).$$

The $\pm$ signs may be paired in 4 ways, giving 4 answers. Choosing both upper signs gives one answer:

$$\cos(\alpha - \beta) = +\frac{4}{5}\left(-\frac{2}{3}\right) + \frac{3}{5}\left(+\frac{\sqrt{5}}{3}\right) = \frac{-8 + 3\sqrt{5}}{15}.$$

The student should write out the other three answers.

EXERCISES 31

By use of the equations in Exercises 1–3 calculate the sine and cosine of the angle on the left.

1. $90° = 135° - 45°$. **2.** $15° = 60° - 45°$. **3.** $105° = 135° - 30°$.

Prove the identities of Exercises 4–9 by use of (c) or (d).

4. $\sin(90° - \alpha) = \cos\alpha$.
7. $\cos(180° - \alpha) = -\cos\alpha$.
5. $\cos(90° - \alpha) = \sin\alpha$.
8. $\sin(270° - \alpha) = -\cos\alpha$.
6. $\sin(180° - \alpha) = \sin\alpha$.
9. $\cos(270° - \alpha) = -\sin\alpha$.

10. Given $\sin\alpha = \frac{12}{13}$ and $\cos\beta = \frac{4}{5}$, α and β in quadrant I, calculate $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$.

11. Given $\sin\alpha = \frac{12}{13}$ and $\cos\beta = \frac{4}{5}$, α in quadrant II and β in quadrant IV, calculate $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$.

12. Prove: $\sin(\alpha - 45°) = \dfrac{\sin\alpha - \cos\alpha}{\sqrt{2}}$; $\cos(\alpha - 45°) = \dfrac{\cos\alpha + \sin\alpha}{\sqrt{2}}$.

68. Formulas for tan $(\alpha \pm \beta)$ and cot $(\alpha \pm \beta)$.

Dividing (a) by (b), member by member, we have

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}.$$

Hence

(e) $$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Similarly,

(f) $$\cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$$

Also, from (e) and (f), by changing the sign of β,

(g) $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

(h) $$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}.$$

Example.

With the data of Example 3, §66, calculate $\tan(\alpha-\beta)$. First calculate $\tan\alpha=\frac{3}{4}$, $\tan\beta=\frac{\sqrt{5}}{2}$. Then

$$\tan(\alpha-\beta)=\frac{\frac{3}{4}-\frac{\sqrt{5}}{2}}{1+\left(\frac{3}{4}\right)\left(\frac{\sqrt{5}}{2}\right)}=\frac{6-4\sqrt{5}}{8+3\sqrt{5}}.$$

EXERCISES 32

Calculate the tangent and cotangent of the first angle in each of the equations below.

1. $15° = 60° - 45°$.
2. $105° = 60° + 45°$.
3. $105° = 135° - 30°$.
4. $165° = 135° + 30°$.
5. $135° = 180° - 45°$.
6. $225° = 180° + 45°$.

69. Formulas, Group B.

For convenience we collect formulas (a), (b) . . . , (h) and form Group B, numbering them consecutively with the formulas of Group A. The formulas for cot $(\alpha \pm \beta)$ may be omitted; in place of them use the formulas for tan $(\alpha \pm \beta)$ with the fractions inverted.

Formulas, Group B

•(9) $$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

(10) $$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$$

(11) $$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$$

(12) $$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

(13) $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

(14) $$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}.$$

(15) $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

(16) $$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}.$$

70. EXERCISES 33

In Exercises 1–8 calculate $\sin\theta$, $\cos\theta$, $\tan\theta$.

1. $\theta = 75° = 45° + 30°.$
2. $\theta = 105° = 150° - 45°.$
3. $\theta = 180° = 150° + 30°.$
4. $\theta = 285° = 240° + 45°.$
5. $\theta = 15° = 45° - 30°.$
6. $\theta = 15° = 150° - 135°.$
7. $\theta = 105° = 240° - 135°.$
8. $\theta = 195° = 240° - 45°.$

9. If $\cos\alpha = \frac{3}{5}$, $\cos\beta = \frac{9}{41}$, α and β in quadrant I, calculate $\cos(\alpha + \beta)$.

10. If $\sin\alpha = \frac{12}{13}$, $\sin\beta = \frac{5}{13}$, α and β in quadrant II, calculate $\cos(\alpha - \beta)$.

11. If $\sin x = \frac{2}{5}$, $\sin y = \frac{4}{5}$, calculate $\sin(x + y)$ and $\tan(x + y)$:
(a) when x is in quadrant I and y in quadrant I;
(b) when x is in quadrant I and y in quadrant II;
(c) when x is in quadrant II and y in quadrant I;
(d) when x is in quadrant II and y in quadrant II.

Show that Exercises 12–21 are identities.

12. $\sin(60° + \alpha) - \sin(60° - \alpha) = \sin\alpha.$

13. $\cos(45° + x) - \cos(45° - x) = -\sqrt{2}\sin x.$

14. $\cos(A - 45°) - \sin(A + 45°) = 0.$

15. $\sin 5x\cos x + \cos 5x\sin x = \sin 6x.$

16. $\cos 3x\cos 2x + \sin 3x\sin 2x = \cos x.$

17. $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan\theta + 1}{1 - \tan\theta}.$

18. $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}.$

19. $\frac{\sin(\alpha - \beta)}{\sin\alpha\,\sin\beta} = \cot\beta - \cot\alpha.$

20. $\frac{\cos(u - v)}{\sin u\,\cos v} = \cot u + \tan v.$

21. $\frac{\tan(45° + \alpha)}{\tan(45° - \alpha)} = \frac{(1 + \tan\alpha)^2}{(1 - \tan\alpha)^2}.$

71. Functions of 2α.

Putting $\beta = \alpha$ in (9), (10) and (13) of Group B, we have

$$\text{(17)} \qquad \sin 2\alpha = 2\sin\alpha\cos\alpha,$$

$$\text{(18)} \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$
$$= 1 - 2\sin^2\alpha,$$
$$= 2\cos^2\alpha - 1.$$

$$\text{(19)} \qquad \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$

For $\cot 2\alpha$ use $\frac{1}{\tan 2\alpha}$. Similarly for $\csc 2\alpha$ and $\sec 2\alpha$.

NOTE. Formula (17) in verbal form is: —

The sine of twice an angle equals twice the sine of the angle times the cosine of the angle.

However, we might put $2\alpha = \beta$, $\alpha = \frac{\beta}{2}$ and obtain

$$\text{(17}'\text{)} \qquad \sin\beta = 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}.$$

In verbal form, this would be:

The sine of an angle equals twice the sine of half the angle times the cosine of half the angle.

The essential thing to notice in formulas (17), (18), (19) is that the angle on the left is twice the angle on the right, or, what amounts to the same thing, that the angle on the right is half the angle on the left.

Examples.

1. From (17) or (17′), $\sin 60° = 2\sin 30° \cos 30°$. Check this.
2. From (18), $\cos 180° = \cos^2 90° - \sin^2 90°$
$= 1 - 2\sin^2 90°$
$= 2\cos^2 90° - 1.$ Check these.
3. From (19) $\tan 120° = \dfrac{2\tan 60°}{1-\tan^2 60°}$. Check this.
4. $\sin 3x = 2\sin\frac{3x}{2}\cos\frac{3x}{2}$. (17)
5. $\cos 6x = 1 - 2\sin^2 3x$. (18)
6. Show that $\dfrac{1+\cos 2\alpha}{\sin 2\alpha} = \cot\alpha$.

$$\frac{1+\cos 2\alpha}{\sin 2\alpha} = \frac{1+(2\cos^2\alpha - 1)}{2\sin\alpha\cos\alpha} \qquad (18), (17)$$
$$= \frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha} = \frac{\cos\alpha}{\sin\alpha} = \cot\alpha.$$

7. Calculate the functions of $2x$ when $\cos x = \frac{3}{5}$.

We first find $\sin x = \pm\frac{4}{5}$ and $\tan x = \pm\frac{4}{3}$.

Then $\sin 2x = 2\sin x\cos x = 2(\pm\frac{4}{5})(\frac{3}{5}) = \pm\frac{24}{25}$.

$\cos 2x = 2\cos^2 x - 1 = 2(\frac{3}{5})^2 - 1 = \frac{18}{25} - \frac{25}{25} = -\frac{7}{25}$.

$$\tan 2x = \frac{2(\pm\frac{4}{3})}{1-(\pm\frac{4}{3})^2} = \frac{\pm\frac{8}{3}}{1-\frac{16}{9}} = \mp\frac{24}{7}.$$

We might also get $\tan 2x$ from $\sin 2x \div \cos 2x$. The other three functions can be obtained by inverting the values just calculated.

Observe that $\cos x = \frac{3}{5}$ means that x may lie in quadrant I or IV. Then $2x$ will lie in quadrant II or III. The upper signs in the answers correspond to $2x$ in quadrant II, the lower signs to $2x$ in quadrant III. Check this by looking up in the table the two *basic angles* (§34) and doubling each of them.

EXERCISES 34

1. Obtain the functions of $60°$ by putting $\alpha = 30°$ in these formulas. Check the results.
2. Check the formulas with $\alpha = 150°$.
3. Check the formulas with $\alpha = -60°$.
4. Check (17) and (18) with $\alpha = 45°$.
5. Prove: $2 \sin 20° \cos 20° = \sin 40°$.
6. Prove: $1 + \cos 80° = 2 \cos^2 40°$.
7. Prove: $\sin^2 50° + \cos 100° = \cos^2 50°$.
8. Prove: $1 - \tan^2 40° = \dfrac{2 \tan 40°}{\tan 80°}$.
9. Calculate the value of $\tan 2x$ when $\tan x = \frac{4}{3}$.
10. Calculate the functions of 2α when $\sin \alpha = \frac{5}{13}$.

Answers: $\sin 2\alpha = \pm \frac{120}{169}$; $\cos 2\alpha = \frac{119}{169}$; $\tan 2\alpha = \pm \frac{120}{119}$.

Prove the following identities.

11. $\sin 4\alpha = 2 \sin 2\alpha \cos 2\alpha$.
12. $\cos 4x = 1 - 2 \sin^2 2x = 1 - 8 \sin^2 x \cos^2 x$.
13. $\dfrac{1 - \cos 2x}{\sin 2x} = \tan x$.
14. $(\sin \beta + \cos \beta)^2 = 1 + \sin 2\beta$.

72. Functions of $\frac{1}{2}\alpha$.

The second and third values of $\cos 2\alpha$ in (18) are

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha, \quad \cos 2\alpha = 2 \cos^2 \alpha - 1.$$

Solving these for $\sin \alpha$ and $\cos \alpha$ respectively, we have

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos 2\alpha}{2}}, \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos 2\alpha}{2}}.$$

Replacing α by $\frac{1}{2}\alpha$, these become

$$\textbf{sin } \tfrac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{2}}, \tag{20}$$

$$\textbf{cos } \tfrac{1}{2}\alpha = \pm \sqrt{\frac{1 + \cos \alpha}{2}}. \tag{21}$$

In formula (20) the sign before the radical must be taken + when angle $\frac{1}{2}\alpha$ lies in quadrant I or II because the sine function is positive in those quadrants; the sign must be taken − when angle $\frac{1}{2}\alpha$ lies in quadrant III or IV.

In formula (21) the sign before the radical must be taken + when $\frac{1}{2}\alpha$ lies in quadrant I or IV, and − when $\frac{1}{2}\alpha$ lies in quadrant II or III.

Dividing (20) by (21), member by member,

$$(22)\qquad \tan \tfrac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}.$$

The second of these forms is obtained by multiplying both sides of the fraction under the radical sign by $1 - \cos\alpha$. This gives

$$\tan \tfrac{1}{2}\alpha = \pm\sqrt{\frac{(1-\cos\alpha)^2}{1-\cos^2\alpha}} = \pm\sqrt{\frac{(1-\cos\alpha)^2}{\sin^2\alpha}} = \frac{1-\cos\alpha}{\sin\alpha}.$$

This fraction always has the same sign as $\tan\frac{1}{2}\alpha$, so the sign $\pm$ has been dropped. The third form for $\tan\frac{1}{2}\alpha$ comes from using the multiplier $1 + \cos\alpha$ instead of $1 - \cos\alpha$.

The student should state formulas (20), (21), (22) in verbal form. Note that the angle on the right is twice the angle on the left.

Examples.

1. In formula (20) put $\alpha = 30°$. Then

$$\sin 15° = +\sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

2. Prove that $\dfrac{1-\sec\alpha}{\sec\alpha} = -2\sin^2\dfrac{\alpha}{2}$.

$$\frac{1-\sec\alpha}{\sec\alpha} = \frac{1}{\sec\alpha} - 1 = \cos\alpha - 1 = \left(1 - 2\sin^2\frac{\alpha}{2}\right) - 1 = -2\sin^2\frac{\alpha}{2}.$$

3. Prove that $\sin\alpha\tan\dfrac{\alpha}{2} = 2\sin^2\dfrac{\alpha}{2}$.

$$\sin\alpha\tan\frac{\alpha}{2} = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\tan\frac{\alpha}{2} = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$= 2\sin^2\frac{\alpha}{2}.$$

73. Formulas, Group C.

(17) $$\sin 2\alpha = 2 \sin \alpha \cos \alpha.$$

(18) $$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1.$$

(19) $$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$$

(20) $$\sin \tfrac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$$

(21) $$\cos \tfrac{1}{2}\alpha = \pm \sqrt{\frac{1 + \cos \alpha}{2}}.$$

(22) $$\tan \tfrac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}.$$

74. EXERCISES 35

1. Calculate the values of $\cos 15°$ and of $\tan 15°$.
2. Calculate the functions of $22\tfrac{1}{2}°$ from those of $45°$.
3. Calculate $\tan \frac{\alpha}{2}$ when $\cos \alpha = \frac{3}{5}$, α in quadrant I.
4. Calculate the values of $\tan 2\alpha$ when $\cos \alpha = \frac{4}{5}$.

Prove the following identities.

5. $\sin 6\alpha = 2 \sin 3\alpha \cos 3\alpha$.
6. $\cos 2\theta = \dfrac{2 - \sec^2 \theta}{\sec^2 \theta}$.
7. $\tan \dfrac{\alpha}{2} = \csc \alpha - \cot \alpha$.
8. $\cot \alpha - \tan \alpha = 2 \cot 2\alpha$.
9. $\tan 2\theta = \dfrac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$.
10. $\sec 2\theta = \dfrac{1}{2 \cos^2 \theta - 1}$.
11. $\cos^4 \beta - \sin^4 \beta = \cos 2\beta$.
12. $\left(\sin \dfrac{\beta}{2} - \cos \dfrac{\beta}{2}\right)^2 = 1 - \sin \beta$.
13. $\sin x \cot \dfrac{x}{2} = 2 \cos^2 \dfrac{x}{2}$.
14. $\cos x(1 + \sec x) = 2 \cos^2 \dfrac{x}{2}$.
15. $2 \tan \alpha \cot 2\alpha = 1 - \tan^2 \alpha$.
16. $2 \cot \dfrac{\alpha}{2} \cot \alpha = \cot^2 \dfrac{\alpha}{2} - 1$.
17. $\cos 2x = \sin^2 x (\cot^2 x - 1)$.
18. $\sec 2x = \dfrac{\csc^2 x}{\cot^2 x - 1}$.

75. Formulas for sin u ± sin v and for cos u ± cos v.

Formulas (9) to (12) of Group B are

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Forming the sum and difference, respectively, of the first two equations, we have

(p) $$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta;$$
(q) $$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta.$$

Forming the sum and difference, respectively, of the other two equations, we have

(r) $$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta;$$
(s) $$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta.$$

Now in the last four equations let

$$\alpha+\beta = u \quad\text{and}\quad \alpha-\beta = v.$$

Then $$\alpha = \frac{u+v}{2} \quad\text{and}\quad \beta = \frac{u-v}{2}.$$

Substituting in equations (p), (q), (r), (s), we have four formulas, called the *addition theorems* of trigonometry, namely

Formulas, Group D

(23) $$\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}.$$

(24) $$\sin u - \sin v = 2\cos\frac{u+v}{2}\sin\frac{u-v}{2}.$$

(25) $$\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}.$$

(26) $$\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}.$$

The four equations (p), (q), (r), (s) are themselves often considered as a group of formulas, and are repeated below, with right and left members interchanged.

NOTE. When u is less than v, the angle in the second factor on the right is negative. Change to the positive angle by use of §23.

Formulas, Group D′

$$(23') \qquad 2 \sin \alpha \cos \beta = \sin (\alpha + \beta) + \sin (\alpha - \beta).$$
$$(24') \qquad 2 \cos \alpha \sin \beta = \sin (\alpha + \beta) - \sin (\alpha - \beta).$$
$$(25') \qquad 2 \cos \alpha \cos \beta = \cos (\alpha + \beta) + \cos (\alpha - \beta).$$
$$(26') \qquad - 2 \sin \alpha \sin \beta = \cos (\alpha + \beta) - \cos (\alpha - \beta).$$

Example 1.

$$\begin{aligned} \sin 60° + \sin 40° &= 2 \sin \frac{60° + 40°}{2} \cos \frac{60° - 40°}{2} \\ &= 2 \sin 50° \cos 10°. \end{aligned}$$

Example 2.

$$\begin{aligned} \sin 60° - \sin 40° &= 2 \cos \frac{60° + 40°}{2} \sin \frac{60° - 40°}{2} \\ &= 2 \cos 50° \sin 10°. \end{aligned}$$

Example 3.

$$\begin{aligned} \sin 40° - \sin 60° &= 2 \cos \frac{40° + 60°}{2} \sin \frac{40° - 60°}{2} \\ &= 2 \cos 50° \sin (- 10°) \\ &= - 2 \cos 50° \sin 10°. \end{aligned}$$

We might also write $\sin 40° - \sin 60° = - (\sin 60° - \sin 40°)$, and proceed as in Example 2.

Example 4.

$$\begin{aligned} 2 \cos 80° \cos 50° &= \cos (80° + 50°) + \cos (80° - 50°) \\ &= \cos 130° + \cos 30°. \end{aligned}$$

Example 5.

$$- 2 \sin 80° \sin 50° = \cos 130° - \cos 30°.$$

Example 6.

Show that $\dfrac{\cos 75° + \cos 15°}{\cos 75° - \cos 15°} = - \sqrt{3}.$

$$\begin{aligned} \frac{\cos 75° + \cos 15°}{\cos 75° - \cos 15°} &= \frac{2 \cos 45° \cos 30°}{- 2 \sin 45° \sin 30°} \\ &= - \cot 45° \cot 30° = - \sqrt{3}. \end{aligned}$$

Example 7.

Show that $\dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.$

$$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\frac{2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}}{2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}} \qquad (23), (24)$$

$$=\tan\frac{\alpha+\beta}{2}\cot\frac{\alpha-\beta}{2}=\frac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.$$

76. EXERCISES 36

Express the sums or differences as products:

1. $\sin 70°+\sin 50° = ?$

2. $\cos 70°+\cos 50° = ?$

3. $\sin 70°-\sin 50° = ?$

4. $\sin 50°-\sin 70° = ?$

5. $\cos 80°-\cos 50° = ?$

6. $\cos 50°+\cos 80° = ?$

7. $\sin 140°+\sin 160° = ?$

8. $\cos 140°-\cos 160° = ?$

9. $\sin 140°+\cos 160° = ?$ (NOTE. $\cos 160° = -\sin 70°$.)

10. $\sin 40°+\cos 70° = ?$

11. $\cos 280°+\sin 140° = ?$

Express the products as sums or differences:

12. $2\sin 60°\cos 20° = ?$

13. $2\cos 60°\sin 20° = ?$

14. $2\cos 60°\cos 20° = ?$

15. $2\sin 60°\sin 20° = ?$

16. $2\cos 130°\sin 50° = ?$

17. $2\cos 40°\cos 140° = ?$

Prove the identities:

18. $\sin 3x+\sin 5x = 2\sin 4x\cos x$.

19. $\sin 10\alpha+\sin 6\alpha = 2\sin 8\alpha\cos 2\alpha$.

20. $\cos 2x+\cos 4x = 2\cos 3x\cos x$.

21. $\sin 7\beta-\sin 5\beta = 2\cos 6\beta\sin\beta$.

22. $\cos 4\theta-\cos 6\theta = 2\sin 5\theta\sin\theta$.

23. $\cos y+\cos 2y = 2\cos\dfrac{3y}{2}\cos\dfrac{y}{2}$.

24. $\cos(\alpha+45°)+\cos(\alpha-45°) = \sqrt{2}\cos\alpha$.

25. $\sin\left(\dfrac{\pi}{3}-x\right)-\sin\left(\dfrac{\pi}{3}+x\right) = -\sin x$.

26. $2\sin 5\alpha\cos 3\alpha = \sin 8\alpha+\sin 2\alpha$.

27. $2\sin 4\theta\sin\theta = \cos 3\theta-\cos 5\theta$.

28. $2\cos\alpha\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta)$.

29. $2\cos\left(\alpha+\dfrac{\pi}{6}\right)\cos\left(\alpha-\dfrac{\pi}{6}\right) = \cos 2\alpha+\tfrac{1}{2}$.

77. EXERCISES 37

These exercises are placed here to afford further drill in the use of the basic formulas of Trigonometry. Many are quite simple; others will test the ingenuity of the best students.

1. If $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{3}{5}$, find the value of $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ when α and β are both in the first quadrant.

2. As in exercise 1, when α and β are both in the second quadrant.

3. If $\cos x = \frac{3}{5}$ and $\cos y = \frac{9}{41}$, calculate $\sin(x+y)$ and $\cos(x+y)$ when x and y are both in the first quadrant. Calculate $\sin 2(x+y)$ and $\cos 2(x+y)$.

4. As in exercise 3, when x and y are both in the fourth quadrant.

5. If $\sin x = \frac{1}{3}$ and $\sin y = \frac{2}{3}$, calculate all values of $\sin(x+y)$ and of $\sin(x-y)$.

6. If $\sin\alpha = \frac{3}{4}$ and $\sin\beta = \frac{3}{5}$, calculate all values of $\cos(\alpha+\beta)$ and of $\cos(\alpha-\beta)$.

7. If $\cos\alpha = \frac{3}{4}$ and $\cos\beta = \frac{2}{5}$, calculate all values of $\tan(\alpha+\beta)$ and of $\tan(\alpha-\beta)$.

8. Calculate $\tan(x+y)$ when $\tan x = \sqrt{3}$ and $\cot y = \sqrt{3}$.

9. Calculate the value of $\tan(2x-y)$ when $\tan x = \frac{4}{3}$ and $\tan y = \frac{12}{5}$.

10. Calculate $\cot(\alpha-\beta)$ when $\tan\alpha = k+1$ and $\tan\beta = k-1$.

11. If $\tan\alpha = \frac{1}{7}$ and $\tan\beta = \frac{2}{11}$, calculate $\tan(2\alpha+\beta)$.

12. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta.$

13. $\dfrac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta} = \cot\alpha\tan\beta + 1.$

14. $\dfrac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1.$

15. $\dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} = \tan\alpha - \tan\beta.$

16. $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}.$

17. $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}.$

18. $\sin 3x = 3\sin x - 4\sin^3 x.$

19. $\cos 3x = 4\cos^3 x - 3\cos x.$

20. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$

21. $\cot 3x = \dfrac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}.$

22. $\tan 4\theta = \dfrac{4\tan\theta(1-\tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}.$

23. $\sqrt{2}\sin(A + 45°) = \sin A + \cos A.$

24. $\sqrt{2}\sin(\theta - 45°) = \sin\theta - \cos\theta.$

25. $\sin(\theta+\varphi)\sin(\theta-\varphi) = \cos^2\varphi - \cos^2\theta.$

26. $\cos(u+v)\cos(u-v) = \cos^2 u - \sin^2 v.$

27. $\cot\left(A - \frac{\pi}{4}\right) = \frac{\cot A + 1}{1 - \cot A}.$

28. $\tan\left(\alpha + \frac{\pi}{3}\right) + \tan\left(\alpha - \frac{\pi}{3}\right) = \frac{8\cot\alpha}{\cot^2\alpha - 3}.$

29. $\sin x \sin(y - z) + \sin y \sin(z - x) + \sin z \sin(x - y) = 0.$

30. $\cos x \sin(y - z) + \cos y \sin(z - x) + \cos z \sin(x - y) = 0.$

31. $\cos(x + y + z) = \cos x \cos y \cos z - \cos x \sin y \sin z$
$- \sin x \cos y \sin z - \sin x \sin y \cos z.$

32. $\dfrac{\sin \frac{5\pi}{12}}{\sin \frac{\pi}{12}} + \dfrac{\cos \frac{5\pi}{12}}{\cos \frac{\pi}{12}} = 4.$

33. $\tan\left(\frac{\pi}{4} + \theta\right)\tan\left(\frac{\pi}{4} - \theta\right) = 1.$

34. $\cos\left(\theta + \frac{\pi}{4}\right) + \sin\left(\theta - \frac{\pi}{4}\right) = 0.$

35. $\cot\left(\theta + \frac{\pi}{4}\right) + \tan\left(\theta - \frac{\pi}{4}\right) = 0.$

36. $\cot\left(\theta - \frac{\pi}{4}\right) + \tan\left(\theta + \frac{\pi}{4}\right) = 0.$

37. $\cot\frac{\pi}{8} + \tan\frac{\pi}{8} = 2\sqrt{2}.$

38. $2\cos\frac{\pi}{8} = \sqrt{2 + \sqrt{2}}.$

39. $\cot\theta - \cot 2\theta = \csc 2\theta.$

40. $\sin 2x = \frac{2\tan x}{1 + \tan^2 x}.$

41. $\sec 2x = \frac{\csc^2 x}{\csc^2 x - 2}.$

42. $\sec^2\theta\cos 2\theta = 1 - \tan^2\theta.$

43. $1 + \tan\theta\tan 2\theta = \sec 2\theta.$

44. $1 - \cos 2x = \tan x \sin 2x.$

45. $\sec 2\theta = \frac{1 + \tan^2\theta}{1 - \tan^2\theta}.$

46. $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta.$

47. $\frac{\sin 2\theta}{1 - \cos 2\theta} = \cot\theta.$

48. $\cot^2\theta - 1 = 2\cot\theta\cot 2\theta.$

49. $2 - \sec^2\theta = \sec^2\theta\cos 2\theta.$

50. $\frac{\cos 2\theta}{1 - \sin 2\theta} = \frac{1 + \tan\theta}{1 - \tan\theta}.$

51. $\frac{\cos 3x}{\cos x} = 2\cos 2x - 1.$

52. $\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}.$

53. $\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta.$

54. $\frac{\tan\theta + \cot\theta}{\cot\theta - \tan\theta} = \sec 2\theta.$

55. $\tan(45^\circ + \varphi) - \tan(45^\circ - \varphi) = 2\tan 2\varphi.$

56. $\frac{\cos^3\varphi - \sin^3\varphi}{\cos\varphi - \sin\varphi} = \frac{2 + \sin 2\varphi}{2}.$

57. $\frac{\cos^5\varphi - \sin^5\varphi}{\cos\varphi - \sin\varphi} = 1 + \frac{1}{2}\sin 2\varphi - \frac{1}{4}\sin^2 2\varphi.$

58. $\frac{\sin x + \cos x}{\cos x - \sin x} = \tan 2x + \sec 2x.$

59. $\sin 2x \tan 2x = \frac{4\tan^2 x}{1 - \tan^4 x}.$

60. $\cos^2\theta + \sin^2\theta\cos 2\varphi = \cos^2\varphi + \sin^2\varphi\cos 2\theta.$

61. $1 + \cos 2(\theta - \varphi)\cos 2\varphi = \cos^2\theta + \cos^2(\theta - 2\varphi).$

62. $\dfrac{\tan^2\left(\theta + \frac{\pi}{4}\right) - 1}{\tan^2\left(\theta + \frac{\pi}{4}\right) + 1} = \sin 2\theta.$

63. $\dfrac{\cos\left(x+\dfrac{\pi}{4}\right)}{\cos\left(x-\dfrac{\pi}{4}\right)} = \sec 2x - \tan 2x.$

64. $\tan x = \dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}.$

65. $\tan x = \dfrac{\sin 2x - \sin x}{1 - \cos x + \cos 2x}.$

66. $\sec 2\theta - \frac{1}{2}\tan 2\theta \sin 2\theta = \dfrac{\cot^2\theta + \tan^2\theta}{\cot^2\theta - \tan^2\theta}.$

67. $\dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \sqrt{\dfrac{1 - \sin 2\theta}{1 + \sin 2\theta}}.$

68. $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$

69. $\left(\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right)^2 = 1 - \sin\theta.$

70. $\dfrac{\cos\theta}{1 + \sin\theta} = \dfrac{1 - \tan\dfrac{\theta}{2}}{1 + \tan\dfrac{\theta}{2}}.$

71. $\dfrac{1 + \tan\dfrac{x}{2}}{1 - \tan\dfrac{x}{2}} = \sec x + \tan x.$

72. $\tan x - \tan\dfrac{x}{2} = \tan\dfrac{x}{2}\sec x.$

73. $\dfrac{1 + \sec\varphi}{\sec\varphi} = 2\cos^2\dfrac{\varphi}{2}.$

74. $\sec^2\dfrac{x}{2} = 2\tan\dfrac{x}{2}\csc x.$

75. $\dfrac{1 + \cos 3\varphi}{\sin 3\varphi} = \cot\dfrac{3\varphi}{2}.$

76. $\dfrac{1 + \sin 45^\circ}{\cos 45^\circ} = \tan 67\frac{1}{2}^\circ.$

77. $\dfrac{1}{\sec\theta + \tan\theta} = \cot\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right).$

78. $\dfrac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot\dfrac{x}{2}.$

79. $\tan\dfrac{x}{2} = \sqrt{\dfrac{2\sin x - \sin 2x}{2\sin x + \sin 2x}}.$

80. $\sqrt{3}\sin 75^\circ - \cos 75^\circ = \sqrt{2}.$

81. $\sin\dfrac{5\theta}{2}\cos\dfrac{\theta}{2} - \sin\dfrac{9\theta}{2}\cos\dfrac{3\theta}{2} + \cos 4\theta\sin 2\theta = 0.$

82. $\sin 4x + \sin 2x = 2\sin 3x\cos x.$

83. $\sin 3x + \sin 5x = 8\sin x\cos^2 x\cos 2x.$

84. $\dfrac{\cot 15^\circ - \tan 15^\circ}{\cot 15^\circ + \tan 15^\circ} = \dfrac{1}{2}\sqrt{3}.$

85. $\dfrac{1 - \sqrt{2}\sin 75^\circ}{1 - \sqrt{2}\cos 75^\circ} = -\cot 60^\circ.$

86. $\cos 100^\circ - \cos 40^\circ = -\cos 20^\circ.$

87. $\sin\left(\dfrac{\pi}{3} + \alpha\right) - \sin\left(\dfrac{\pi}{3} - \alpha\right) = \sin\alpha.$

88. $\cos\left(\dfrac{\pi}{4} + \alpha\right) - \cos\left(\dfrac{\pi}{4} - \alpha\right) = -\sqrt{2}\sin\alpha.$

89. $\cos(\theta + \varphi) + \sin(\theta - \varphi) = 2\cos\left(\dfrac{\pi}{4} - \theta\right)\cos\left(\dfrac{\pi}{4} + \varphi\right).$

90. $2\sin\left(\alpha + \dfrac{\pi}{4}\right)\sin\left(\alpha - \dfrac{\pi}{4}\right) = \sin^2\alpha - \cos^2\alpha.$

91. $\sin\left(\dfrac{\pi}{4} + \alpha\right) - \sin\left(\dfrac{\pi}{4} - \alpha\right) = \sqrt{2}\sin\alpha.$

92. $\cos 3x - \cos x = -4\sin^2 x\cos x.$

93. $\dfrac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}.$

94. $\dfrac{\cos x + \cos y}{\cos x - \cos y} = -\cot\dfrac{x+y}{2}\cot\dfrac{x-y}{2}.$

95. $\dfrac{(\sin\alpha + \sin\beta)(\cos\alpha + \cos\beta)}{(\sin\alpha - \sin\beta)(\cos\alpha - \cos\beta)} = -\cot^2\dfrac{\alpha - \beta}{2}.$

96. $\dfrac{(\sin\alpha + \sin\beta)(\cos\alpha - \cos\beta)}{(\sin\alpha - \sin\beta)(\cos\alpha + \cos\beta)} = -\tan^2\dfrac{\alpha + \beta}{2}.$

97. $\dfrac{(\sin 75^\circ + \sin 15^\circ)(\cos 75^\circ + \cos 15^\circ)}{(\sin 75^\circ - \sin 15^\circ)(\cos 75^\circ - \cos 15^\circ)} = -3.$

98. $\dfrac{\cos 2x + \cos 12x}{\cos 6x + \cos 8x} + \dfrac{\cos 7x - \cos 3x}{\cos x - \cos 3x} + \dfrac{2\sin 4x}{\sin 2x} = 0.$

99. $\sin x + \sin 2x + \sin 3x = 4\cos\frac{1}{2}x\cos x\sin\frac{3}{2}x.$

(*Hint.* Replace $\sin x + \sin 3x$ by $2\sin 2x\cos x$ and $\sin 2x$ by $2\sin x\cos x$; from these results factor out $2\cos x$ and combine the remainders by the formula for $\sin u + \sin v$.)

100. $\sin x - \sin 2x + \sin 3x = 4\sin\frac{1}{2}x\cos x\cos\frac{3}{2}x.$

101. $\cos x - \cos 2x + \cos 3x = 1 - 4\sin\frac{1}{2}x\cos x\sin\frac{3}{2}x.$

102. $\dfrac{\sin\theta + \sin 2\theta + \sin 3\theta}{\cos\theta + \cos 2\theta + \cos 3\theta} = \tan 2\theta.$

103. $\cos 20^\circ + \cos 100^\circ - \cos 140^\circ = 0.$

104. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4\cos\theta\cos 2\theta\cos 4\theta.$

105. $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 16\sin\theta\cos^2\theta\cos^2 2\theta.$

106. $4\sin^2\varphi\cos^2\varphi + (\cos^2\varphi - \sin^2\varphi)^2 = 1.$

107. $(\cos x\cos y + \sin x\sin y)^2 + (\sin x\cos y - \cos x\sin y)^2 = 1.$

108. $\dfrac{\tan 3x - \tan x}{1 + \tan 3x\tan x} = \tan 2x.$

109. $\dfrac{\tan(n+1)\theta - \tan n\theta}{1 + \tan(n+1)\theta\tan n\theta} = \tan\theta.$

110. $\dfrac{\tan(\theta + \varphi) - \tan\varphi}{1 + \tan(\theta + \varphi)\tan\varphi} = \tan\theta.$

111. $\dfrac{\tan(\theta - \varphi) + \tan\varphi}{1 - \tan(\theta - \varphi)\tan\varphi} = \tan\theta.$

112. $\sin n\theta \cos\theta + \cos n\theta \sin\theta = \sin(n+1)\theta.$

113. $2\csc 4x + 2\cot 4x = \cot x - \tan x.$

114. If $\tan x = \dfrac{b}{a}$, show that $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}} = \dfrac{2\cos x}{\sqrt{\cos 2x}}.$

115. $4\cos^3 x \sin 3x + 4\sin^3 x \cos 3x = 3\sin 4x.$ (See Ex's 18, 19.)

116. $\sin^3 x + \sin^3(120° + x) + \sin^3(240° + x) = -\frac{3}{4}\sin 3x.$

117. $\cos 6x = 16(\cos^6 x - \sin^6 x) - 15\cos 2x.$

118. $1 + \tan^6 x = \sec^4 x(\sec^2 x - 3\sin^2 x).$

119. $\dfrac{3\sin x - \sin 3x}{3\cos x + \cos 3x} = \tan^3 x.$ (See Ex's 18, 19.)

120. $\sin 2x \sin 2y = \sin^2(x+y) - \sin^2(x-y).$ (Factor the right-hand side.)

121. $\sin 5\alpha \sin\alpha = \sin^2 3\alpha - \sin^2 2\alpha.$

122. $8\cos^2\alpha - 1 + \cos 4\alpha = 8\cos^4\alpha.$

123. $\cos 2x + \cos 2y + \cos 2z + \cos 2(x+y+z)$
$= 4\cos(x+y)\cos(y+z)\cos(z+x).$

124. $\sin^2 x + \sin^2 y + \sin^2 z + \sin^2(x+y+z)$
$= 2 - 2\cos(x+y)\cos(y+z)\cos(z+x).$

125. $\cos^2 x + \cos^2 y + \cos^2 z + \cos^2(x+y-z)$
$= 2 + 2\cos(x+y)\cos(x-z)\cos(y-z).$

126. $\sin(x-y-z) - \sin x - \sin y - \sin z = 4\sin\dfrac{x-y}{2}\sin\dfrac{x-z}{2}\sin\dfrac{y+z}{2}.$

127. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \sin 2(\alpha+\beta+\gamma) + 4\sin(\alpha+\beta)\sin(\beta+\gamma)$
$\sin(\alpha+\gamma).$

128. $\sin(\alpha+\beta-\gamma) + \sin(\alpha-\beta+\gamma) + \sin(\beta+\gamma-\alpha) - \sin(\alpha+\beta+\gamma)$
$= 4\sin\alpha\sin\beta\sin\gamma.$

129. $\cos(\alpha+\beta-\gamma) + \cos(\beta+\gamma-\alpha) + \cos(\alpha+\gamma-\beta) + \cos(\alpha+\beta+\gamma)$
$= 4\cos\alpha\cos\beta\cos\gamma.$

130. Show that the equation $\sin x = a + \dfrac{1}{a}$ is impossible.

131. For what values of a will the equation $2\cos x = a + \dfrac{1}{a}$ give possible values for x? *Ans.* $a = \pm 1.$

CHAPTER

VIII

OBLIQUE PLANE TRIANGLES

78. *The law of sines.*

Between the six parts of a plane triangle there exist, aside from the angle-sum equal to 180°, two other fundamental relations which we proceed to obtain. Additional relations will then be derived from these.

In any plane triangle, the sides are proportional to the sines of the opposite angles.

Let ABC be the triangle, CD one of its altitudes. Two cases arise, according as D falls within or without the base (figures).

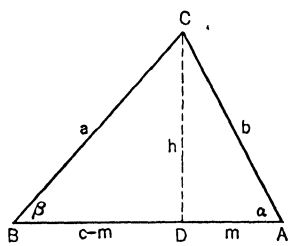


FIG. 59a

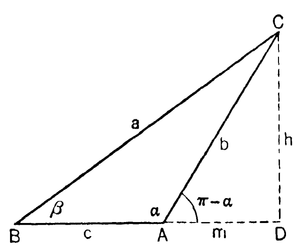


FIG. 59b

	First figure	*Second figure.*
From $\triangle\ ACD$,	$h = b \sin \alpha$;	$h = b \sin (\pi - \alpha) = b \sin \alpha$.
From $\triangle\ BCD$,	$h = a \sin \beta$;	$h = a \sin \beta$.

Equating the values of h, we have in either case

$$b \sin \alpha = a \sin \beta, \quad \text{or} \quad \frac{a}{\sin \alpha} = \frac{b}{\sin \beta}.$$

By drawing perpendiculars from the other vertices and combining results we have the ***law of sines,***

(1) $$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

79. The law of cosines.

In any plane triangle, the square of any side equals the sum of the squares of the other two sides, minus twice their product by the cosine of their included angle.

In the above figures let $AD = m$.

	First figure	*Second figure.*
In $\triangle ACD$,	$m = b \cos \alpha;$	$m = b \cos (\pi - \alpha) = - b \cos \alpha.$
In $\triangle BCD$,	$a^2 = h^2 + (c - m)^2$	$a^2 = h^2 + (c + m)^2$
	$= h^2 + c^2 - 2cm + m^2.$	$= h^2 + c^2 + 2cm + m^2.$

But, in either figure, $h^2 + m^2 = b^2.$

Hence $a^2 = b^2 + c^2 - 2cm.$ $\qquad a^2 = b^2 + c^2 + 2cm.$

Replacing m by its value above, we have in either case,

(2) $$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

(2′) Similarly, $$b^2 = a^2 + c^2 - 2ac \cos \beta,$$

(2″) and, $$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

The verbal statement of the law of cosines covers all three of these equations.

80. Applications of the law of sines and the law of cosines.

Example 1.

In $\triangle ABC$, given $a = 40$, $b = 35$, $\alpha = 50°$; to determine angle β to the nearest minute.

Law of sines: $\frac{a}{b} = \frac{\sin \alpha}{\sin \beta}$ or $\sin \beta = \frac{b}{a} \sin \alpha.$

Substitute the given values:

$$\sin \beta = \tfrac{35}{40} \sin 50° = \tfrac{7}{8} \times 0.7660 = 0.6702. \qquad \text{(Table III)}$$

The basic angles (§34) are: $\beta = 42° 5'$; $\beta' = 137° 55'$. We have two possible values for angle β, but the second value must be discarded as impossible because the sum $\alpha + \beta' = 50° + 137° 55'$ exceeds 180°.

Fig. 60 shows the triangle drawn to scale, one marked segment representing 5 units of length. First construct an angle of $50° = \alpha$; on one of the sides of α lay off $b = 35 = AC$. With C as center and radius $a = 40$ strike an arc to cut the second side of angle α.

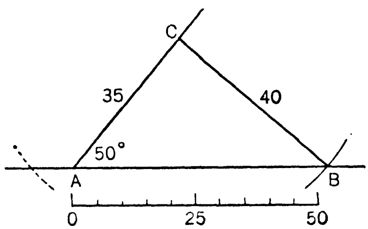


FIG. 60

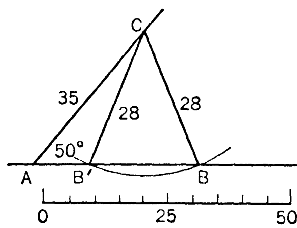


FIG. 61

Example 2.

In $\triangle ABC$, given $a = 28$, $b = 35$, $\alpha = 50°$; to determine angle β to the nearest minute.

As in Example 1: $\sin \beta = \frac{35}{28} \sin 50° = 0.9575.$

Basic angles: $\beta = 73° 14'$; $\beta' = 106° 46'$.

Fig. 61 shows the construction and indicates two possible triangles: $\triangle ABC$ with basic angle $\beta = \angle ABC$ and $\triangle AB'C$ with basic angle $\beta' = \angle AB'C$.

Example 3.

Two sides of a parallelogram are 40 ft. and 50 ft. long, respectively, and their included angle is 50°. Determine the length of the shorter diagonal. (Figure 62.)

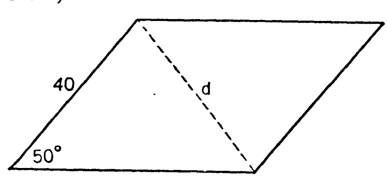


FIG. 62

By the law of cosines:

$$d^2 = 40^2 + 50^2 - 2 \cdot 40 \cdot 50 \cos 50°$$
$$= 1600 + 2500 - 4000 \times 0.6428 = 1529.$$
$$d = 39.1+ \text{ feet.}$$

EXERCISES 38

In $\triangle ABC$ calculate the required element. Draw figures to scale.

1. $a = 30,\ b = 25,\ \alpha = 40°;\quad \beta = ?$
2. $a = 20,\ b = 25,\ \alpha = 40°;\quad \beta = ?$
3. $b = 100,\ c = 75,\ \beta = 45°;\quad \gamma = ?$
4. $b = 75,\ c = 100,\ \beta = 45°;\quad \gamma = ?$
5. $a = 75,\ c = 90,\ \gamma = 55°;\quad \alpha = ?$
6. $a = 90,\ c = 75,\ \gamma = 55°;\quad \alpha = ?$
7. $a = 5,\ b = 6,\ \gamma = 70°;\quad c = ?$
8. $a = 10,\ b = 15,\ \gamma = 45°;\quad c = ?$
9. $a = 25,\ c = 40,\ \beta = 60°;\quad b = ?$
10. $a = 30,\ c = 100,\ \beta = 30°;\quad b = ?$
11. $a = 4,\ b = 5,\ c = 7;\quad \alpha, \beta, \gamma = ?$
12. $a = 10,\ b = 15,\ c = 20;\quad \alpha, \beta, \gamma = ?$
13. $a = 30,\ b = 25,\ c = 20;\quad \alpha, \beta, \gamma = ?$
14. In Example 3 calculate the long diagonal.
15. An airplane travels E 40° N a distance of 150 miles, then E 70° N a distance of 200 miles. How far is it now from the starting point? Solve by the law of cosines.

81. The law of tangents.

In any plane triangle, the difference of two sides is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

From the law of sines: $\dfrac{a}{b} = \dfrac{\sin \alpha}{\sin \beta}.$

Therefore: $\dfrac{a}{b} + 1 = \dfrac{\sin \alpha}{\sin \beta} + 1$ and $\dfrac{a}{b} - 1 = \dfrac{\sin \alpha}{\sin \beta} - 1.$

Therefore: $\dfrac{a+b}{b} = \dfrac{\sin \alpha + \sin \beta}{\sin \beta}$ and $\dfrac{a-b}{b} = \dfrac{\sin \alpha - \sin \beta}{\sin \beta}.$

Dividing the last equation by the preceding equation gives

$$\begin{aligned}\frac{a-b}{a+b} &= \frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta}\\ &= \frac{2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)}{2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)}\\ &= \cot \tfrac{1}{2}(\alpha + \beta) \tan \tfrac{1}{2}(\alpha - \beta).\end{aligned}$$

That is,

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}. \tag{3}$$

Similarly,

(3′) $$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)},$$

(3″) and $$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}.$$

The symmetry of these formulas makes them easy to remember. In actual practice, they are used in slightly modified form. Thus the first of them is written,

$$\tan\tfrac{1}{2}(\alpha-\beta) = \frac{a-b}{a+b}\tan\tfrac{1}{2}(\alpha+\beta).$$

Example.

In $\triangle ABC$, $a = 15$, $b = 10$, $\gamma = 50°$. Determine the angles α, β to the nearest minute.

Substitute in (3):

$$a - b = 5,\ a + b = 25,\ \tfrac{1}{2}(\alpha+\beta) = \tfrac{1}{2}(180° - 50°) = 65°.$$

Then: $\dfrac{5}{25} = \dfrac{\tan\frac{1}{2}(\alpha-\beta)}{\tan 65°}$; $\tan\frac{1}{2}(\alpha-\beta) = \frac{1}{5}\tan 65° = 0.4289.$

$\frac{1}{2}(\alpha-\beta) = 23°\ 13'$; sum $= \alpha = 88°\ 13'$,

$\frac{1}{2}(\alpha+\beta) = 65°$; difference $= \beta = 41°\ 47'$.

EXERCISES

In $\triangle ABC$ determine the two angles not given.

1. $a = 25$, $b = 15$, $\gamma = 60°$.
2. $b = 16$, $c = 12$, $\alpha = 40°$.
3. $a = 50$, $c = 25$, $\beta = 42°$.
4. $a = 24$, $b = 36$, $\gamma = 70°$.

82. Functions of the half-angles.

When the three sides of a triangle are known, its angles are best calculated by the formulas now to be derived.

From the law of cosines we have,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}.$$

In practice this formula is not convenient unless a, b, and c happen to be simple numbers. Now

$$\sin\tfrac{1}{2}\alpha = \sqrt{\frac{1-\cos\alpha}{2}}.\quad \left(\text{Why not } \pm\sqrt{\frac{1-\cos\alpha}{2}}?\right)$$

But $1 - \cos\alpha = 1 - \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{2bc - b^2 - c^2 + a^2}{2bc}$

$$= \frac{a^2 - (b - c)^2}{2bc} = \frac{[a + (b - c)][a - (b - c)]}{2bc}.$$

$$\frac{1 - \cos\alpha}{2} = \frac{(a + b - c)(a - b + c)}{4bc}.$$

Let $2s = a + b + c$, or $s = \frac{1}{2}(a + b + c)$.

Then $2(s - c) = a + b - c$, and $2(s - b) = a - b + c$.

Hence
$$\frac{1 - \cos\alpha}{2} = \frac{2(s - b)2(s - c)}{4bc}$$

and, taking square roots,

(4) $$\boldsymbol{\sin \tfrac{1}{2}\alpha = \sqrt{\frac{(s - b)(s - c)}{bc}}.}$$

Similarly,

(4′) $$\boldsymbol{\sin \tfrac{1}{2}\beta = \sqrt{\frac{(s - a)(s - c)}{ac}},}$$

(4″) and $$\boldsymbol{\sin \tfrac{1}{2}\gamma = \sqrt{\frac{(s - a)(s - b)}{ab}}.}$$

Observe that the sides appearing explicitly under the radical *include* the angle to be calculated.

To obtain $\cos \frac{1}{2}\alpha$, we have

$$\cos \tfrac{1}{2}\alpha = \sqrt{\frac{1 + \cos\alpha}{2}}.$$

But
$$1 + \cos\alpha = 1 + \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc}$$

$$= \frac{(b + c + a)(b + c - a)}{2bc}$$

$$= \frac{4s(s - a)}{2bc}.$$

Hence

(5) $$\boldsymbol{\cos \tfrac{1}{2}\alpha = \sqrt{\frac{s(s - a)}{bc}}.}$$

Similarly,

$$(5') \qquad \cos \tfrac{1}{2}\beta = \sqrt{\frac{s(s-b)}{ac}},$$

$$(5'') \text{ and } \qquad \cos \tfrac{1}{2}\gamma = \sqrt{\frac{s(s-c)}{ab}}.$$

Dividing sine by cosine we have

$$(6) \qquad \tan \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$(6') \qquad \tan \tfrac{1}{2}\beta = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}},$$

$$(6'') \qquad \tan \tfrac{1}{2}\gamma = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

In (6) multiply both numerator and denominator of the fraction by $s - a$. Then

$$\tan \tfrac{1}{2}\alpha = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Also let
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Then:

$$(7) \qquad \tan \tfrac{1}{2}\alpha = \frac{r}{s-a},$$

$$(7') \qquad \tan \tfrac{1}{2}\beta = \frac{r}{s-b},$$

$$(7'') \qquad \tan \tfrac{1}{2}\gamma = \frac{r}{s-c}.$$

All these formulas should be memorized in *verbal form,* so that a single statement contains all three formulas of any one set.

83. Mollweide's equation.

This is an equation which involves all six parts of triangle ABC and may be used as a check formula to insure that calculated parts of the triangle are correct. The derivation of the equation follows.

$$\frac{a}{c} = \frac{\sin \alpha}{\sin \gamma}; \quad \frac{b}{c} = \frac{\sin \beta}{\sin \gamma}. \qquad \text{(Law of sines.)}$$

$$\frac{a-b}{c} = \frac{\sin\alpha - \sin\beta}{\sin\gamma},$$

$$= \frac{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma}. \qquad (\S 71, \S 75)$$

But $\frac{1}{2}(\alpha+\beta) = 90^\circ - \frac{1}{2}\gamma$ and $\cos\frac{1}{2}(\alpha+\beta) = \sin\frac{1}{2}\gamma$. (§12)
Therefore, on cancelling equal factors, we have Mollweide's equation:

$$\frac{a-b}{c} = \frac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}\gamma}.$$

84. Solution of plane oblique triangles.

A triangle is determined, except in such cases as will be specially mentioned, when three parts are given, of which one at least must be a side. The calculation of the other parts is called "solving the triangle."

Four cases arise, according to the nature of the given parts.

I. *Given one side and two angles.*
II. *Given two sides and their included angle.*
III. *Given two sides and an opposite angle.*
IV. *Given three sides.*

The method for treating each case will now be considered.

85. Case I. Given one side and two angles, as α, β, a.

Formulas for finding the other parts, γ, b, c.

$$\gamma = 180^\circ - (\alpha+\beta).$$

From the law of sines,

$$b = a\frac{\sin\beta}{\sin\alpha}; \quad c = a\frac{\sin\gamma}{\sin\alpha}.$$

Check. It is important to have a check on the accuracy of the calculated parts. For this purpose use a formula not used in the computations and involving as many as possible of these parts.

In this case we use the law of tangents in the form:

$$(b+c)\tan\tfrac{1}{2}(\beta-\gamma) = (b-c)\tan\tfrac{1}{2}(\beta+\gamma).$$

We might also use Mollweide's equation.

Example.

Given $a = 400$, $\alpha = 50°$, $\beta = 100°$. To find b, c, γ.

Graphic solution.

This will give us a fair idea of what answers to expect. First calculate $\gamma = 180° - (50° + 100°) = 30°$. Lay off a line segment equal to a and at its extremities construct angles β and γ, prolonging their free sides to meet at A (figure). Scale off the lengths of b and c. We find $b = 520$ and $c = 260$ approximately.

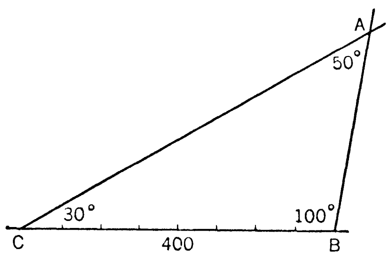


FIG. 63

Logarithmic solution.

Formulas.

$$\gamma = 180° - (\alpha + \beta).$$

$$b = a\frac{\sin\beta}{\sin\alpha}; \quad \log b = \log a + \log\sin\beta - \log\sin\alpha.$$

$$c = a\frac{\sin\gamma}{\sin\alpha}; \quad \log c = \log a + \log\sin\gamma - \log\sin\alpha.$$

Check. $(b + c)\tan\frac{1}{2}(\beta - \gamma) = (b - c)\tan\frac{1}{2}(\beta + \gamma).$

$$\log(b + c) + \log\tan\tfrac{1}{2}(\beta - \gamma) = \log(b - c) + \log\tan\tfrac{1}{2}(\beta + \gamma).$$

The detailed solution follows. Four-place tables are used.

Given: $a = 400$, $\alpha = 50°$, $\beta = 100°$.

Angle γ.

$$\begin{aligned} \alpha &= 50°. \\ \beta &= 100° \\ \hline \alpha + \beta &= 150°. \end{aligned} \qquad 180° - 150° = \gamma = \mathbf{30°}.$$

Side b.		*Side c.*	
$\log a =$	2.6021	$\log a =$	2.6021
$\log\sin\beta =$	9.9934–10	$\log\sin\gamma =$	9.6990–10
	12.5955–10		12.3011–10
$\log\sin\alpha =$	9.8843–10	$\log\sin\alpha =$	9.8843–10
$\log b =$	2.7112	$\log c =$	2.4168
$\boldsymbol{b} =$	**514.3.**	$\boldsymbol{c} =$	**261.1.**

Check.

$b + c = 775.4$	$b - c = 253.2$
$\beta - \gamma = 70°$	$\beta + \gamma = 130°$
$\frac{1}{2}(\beta - \gamma) = 35°$	$\frac{1}{2}(\beta + \gamma) = 65°$
$\log (b + c) = 2.8895$	$\log (b - c) = 2.4034$
$\log \tan \frac{1}{2}(\beta - \gamma) = 9.8452\text{–}10$	$\log \tan \frac{1}{2}(\beta + \gamma) = 0.3313$
$\text{sum} = 2.7347.$	$\text{sum} = 2.7347.$

EXERCISES 39

1. $a = 1000$, $\alpha = 50°$, $\beta = 75°$.
2. $a = 5.257$, $\alpha = 62° 35'$, $\beta = 70° 43'$.
3. $b = 7.918$, $\beta = 77° 10'$, $\gamma = 64° 50'$.
4. $c = 0.00835$, $\beta = 121° 35'$, $\gamma = 35° 41'$.
5. $c = 3708$, $\beta = 59° 5'$, $\gamma = 33° 15'$.
6. $b = 15.285$, $\alpha = 130° 18.3'$, $\gamma = 22° 35.2'$.

In the figure of §47 calculate AD and BD from the following data.

7. $m = 350$ ft., $\alpha = 40°$, $\beta = 70°$.
8. $m = 228.3$ ft., $\alpha = 27° 33'$, $\beta = 41° 7'$.
9. $m = 744.7$ ft., $\alpha = 37° 45.3'$, $\beta = 81° 21.6'$.

10. In Exercise 15 of §49 find the distance from each point of observation to the top of the tower.

11. In Exercise 16 of §49 find the distance from each point of observation to the top of the tree.

12. In Exercise 3 of §56 calculate the distance from ship to lighthouse at the time of each observation.

86. Case II. Given two sides and the included angle, as a, b, γ.

To solve the triangle we calculate $\frac{1}{2}(\alpha + \beta)$ as the complement of $\frac{1}{2}\gamma$; then $\frac{1}{2}(\alpha - \beta)$ is calculated by formula (3). Angles α and β are then determined and hence all the angles are known. We can then compute c in two ways by means of the law of sines. The agreement of the two values of c furnishes a check on the computations.

Formulas.

$$\tfrac{1}{2}(\alpha + \beta) = 90° - \tfrac{1}{2}\gamma.$$

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(\alpha + \beta).$$

$$c = a\frac{\sin \gamma}{\sin \alpha} = b\frac{\sin \gamma}{\sin \beta}. \qquad \text{Check.}$$

Check. Duplicate calculation of side c.
Or use Mollweide's equation.

Example.
Given $b = 12.553$, $a = 20.635$, $\gamma = 27° 24.2'$. Solve the triangle.

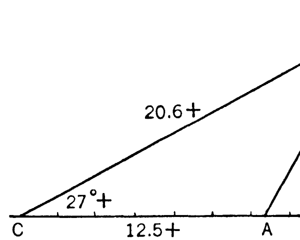


FIG. 64

Graphic solution.

Construct angle γ and on its sides lay off lengths a and b, starting from the vertex. Complete the triangle, and measure c, α, and β. We obtain $c = 11.0$, $\alpha = 119°$, $\beta = 33°$. A solution is possible provided $0 < \gamma < 180°$.

Logarithmic solution.

Formulas.

$$\tfrac{1}{2}(\alpha + \beta) = 90° - \tfrac{1}{2}\gamma.$$
$$\log \tan \tfrac{1}{2}(\alpha - \beta) = \log (a - b) - \log (a + b) + \log \tan \tfrac{1}{2}(\alpha + \beta).$$
$$\log c = \log a + \log \sin \gamma - \log \sin \alpha.$$
$$\log c = \log b + \log \sin \gamma - \log \sin \beta.$$

The detailed solution follows. Five-place tables are used.

Angles α and β.

$$\gamma = 27° 24.2'. \qquad \tfrac{1}{2}\gamma = 13° 42.1'.$$
$$\tfrac{1}{2}(\alpha + \beta) = 90° - 13° 42.1' = 76° 17.9'.$$

$$\begin{array}{rlrl}
a = & 20.635 & \log (a - b) = & 10.90752\text{–}10 \\
b = & \underline{12.553} & \log (a + b) = & \underline{1.52098} \\
a + b = & 33.188 & \text{diff.} = & 9.38654\text{–}10 \\
a - b = & 8.082 & \log \tan \tfrac{1}{2}(\alpha + \beta) = & \underline{0.61295} \\
 & & \log \tan \tfrac{1}{2}(\alpha - \beta) = & 9.99949\text{–}10
\end{array}$$

$$\tfrac{1}{2}(\alpha + \beta) = 76° 17.9' \qquad \alpha = \mathbf{121° 15.9'}.$$
$$\tfrac{1}{2}(\alpha - \beta) = 44° 58.0' \qquad \beta = \mathbf{31° 19.9'}.$$

Side c and check.

$$\begin{array}{rl}
\log a = & 1.31460 \\
\log \sin \gamma = & \underline{9.66300-10} \\
\text{sum} = & 10.97760-10 \\
\log \sin \alpha = & \underline{9.93185-10} \\
\log c = & 1.04575
\end{array}
\qquad
\begin{array}{rl}
\log b = & 1.09874 \\
\log \sin \gamma = & \underline{9.66300-10} \\
\text{sum} = & 10.76174-10 \\
\log \sin \beta = & \underline{9.71600-10} \\
\log c = & 1.04574
\end{array}$$

$$c = \mathbf{11.111}.$$

Caution. Agreement of the two values of c is not a complete check; they may agree, yet both be wrong, due to an error in $\log \sin \gamma$; check this very carefully.

Check by Mollweide's equation.

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}\gamma}, \quad \text{or,} \quad (a-b)\cos \tfrac{1}{2}\gamma = c \sin \tfrac{1}{2}(\alpha - \beta).$$

$$\begin{array}{rl}
\log (a-b) = & 0.90752 \\
\log \cos \frac{1}{2}\gamma = & \underline{9.98746-10} \\
\text{sum} = & 0.89498
\end{array}
\qquad
\begin{array}{rl}
\log c = & 1.04575 \\
\log \sin \frac{1}{2}(\alpha - \beta) = & \underline{9.84923-10} \\
\text{sum} = & 0.89498
\end{array}$$

NOTE. If side b were greater than side a, the difference $a - b$ would be negative, as also the difference $\alpha - \beta$. To avoid negative differences in such cases, interchange letters in the formula for the law of tangents, and write it

$$\tan \tfrac{1}{2}(\beta - \alpha) = \frac{b-a}{b+a} \tan \tfrac{1}{2}(\beta + \alpha).$$

EXERCISES 40

Solve the following triangles:

1. $a = 800,\quad b = 895,\quad \gamma = 60°.$

2. $a = 25.45,\quad c = 21.60,\quad \beta = 52° 30'.$

3. $a = 223,\quad b = 402,\quad \gamma = 101° 40'.$

4. $b = 3124,\quad c = 8976,\quad \alpha = 125° 32'.$

5. $b = .04544,\quad c = .06400,\quad \alpha = 36° 08'.$

6. $a = 541.83,\quad c = 327.68,\quad \beta = 78° 43.7'.$

7. Apply the methods of this section to solve $\triangle ABC$ of Fig. 43, §52, using the data there given.

8. Similarly solve $\triangle ABD$ of Fig. 46, §53.

9. An angle of a triangle is 40° and one of the including sides is twice as long as the other. Determine the other two angles. Check by the law of sines.

10. The difference of two of the sides of a triangle is 50 and the difference of their opposite angles is 30°. The third angle 60°. Solve the triangle.

87. Case III. Given two sides and an opposite angle, as a, b, α.

This is known as the *ambiguous case.* We begin by studying the ***Graphic Solution.*** Lay off angle α and on one of its sides take $AC = b$. With C as center and radius equal to a, strike an arc of a circle. The figures show the various possibilities arising in the construction, the first three for $\alpha < 90°$, the last three for $\alpha > 90°$.

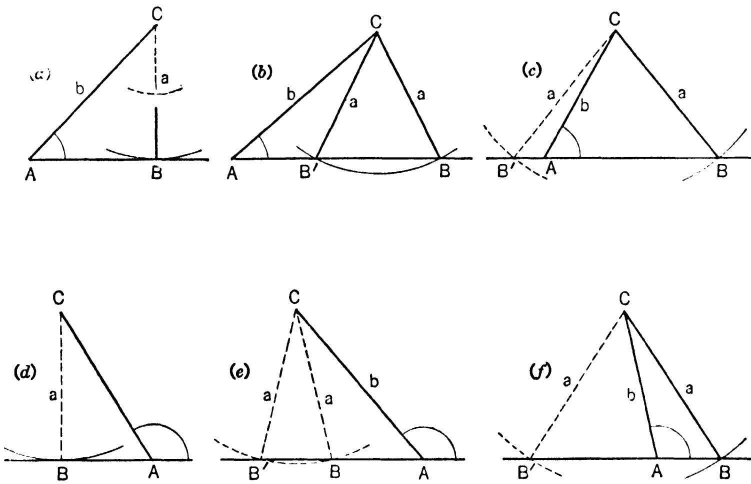


FIG. 65

In each case the perpendicular from C on the other side of angle α is equal to $b \sin \alpha$. Inspection of the figures then shows that

when $\alpha < 90°$ and $a < b \sin \alpha$, no triangle is possible;
when $\alpha < 90°$ and $a = b \sin \alpha$, a right triangle results;
when $\alpha < 90°$ and $b > a > b \sin \alpha$, two oblique triangles result;
when $\alpha < 90°$ and $a \geq b$, one oblique triangle results;
when $\alpha > 90°$ and $a \leq b$, no solution is possible;
when $\alpha > 90°$ and $a > b$, one oblique triangle results.

It is always possible therefore to state in advance what the nature of the solution in a given case will be.

In a given numerical example the nature of the solution always becomes apparent during the progress of the computations.

Formulas. Given a, b, α.

$$\sin \beta = \frac{b}{a}\sin \alpha. \qquad \gamma = 180^\circ - (\alpha + \beta). \quad c = a\frac{\sin \gamma}{\sin \alpha} = b\frac{\sin \gamma}{\sin \beta}.$$

$$\beta' = 180^\circ - \beta. \quad \gamma' = 180^\circ - (\alpha + \beta'). \quad c' = a\frac{\sin \gamma'}{\sin \alpha} = b\frac{\sin \gamma'}{\sin \beta'}.$$

Check. The agreement of the values of c and c' as calculated from the two expressions for each of them furnishes a partial check on the calculations. It does not guard against an error in log sin γ, which may be checked independently. A more positive check is furnished by the law of tangents or by Mollweide's equation.

In carrying out the calculations according to the formulas above, the various cases shown in the figures are indicated as follows:

(a) log sin $\beta \geqq 0$; no solution, or right triangle.
(b) retain both β and β'; two solutions.
(c) $\alpha + \beta' > 180^\circ$, hence reject β'; one solution.
(d) log sin $\beta \geqq 0$; no solution.
(e) $\alpha + \beta > 180^\circ$ and $\alpha + \beta' > 180^\circ$; no solution.
(f) As in (c); one solution.

Example.

Given $a = 602.3$, $b = 764.1$, $\alpha = 38^\circ\ 17'$.

Graphic solution.

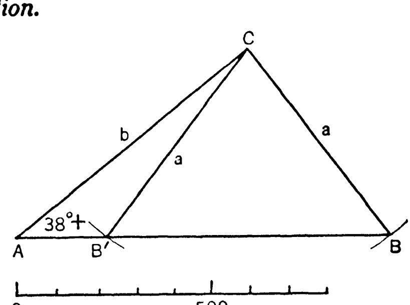


FIG. 66

This is shown in the figure, from which the unknown parts may be scaled off.

Logarithmic solution.
Formulas.

$$\log \sin \beta = \log b - \log a + \log \sin \alpha. \quad \beta = ? \quad \beta' = ?$$

$$\gamma = 180° - (\alpha + \beta). \quad \gamma' = 180° - (\alpha + \beta').$$

$$\begin{aligned} \log c &= \log a + \log \sin \gamma - \log \sin \alpha, \\ &= \log b + \log \sin \gamma - \log \sin \beta. \\ \log c' &= \log a + \log \sin \gamma' - \log \sin \alpha, \\ &= \log b + \log \sin \gamma' - \log \sin \beta'. \end{aligned}$$

Check. Use duplicate calculation of side c.

The detailed solution follows. Four-place tables are used.

Angles β, β', γ, γ'.

$$\begin{aligned} \log b &= 2.8832 \\ \log a &= \underline{2.7798} \\ \text{diff.} &= 0.1034 \\ \log \sin \alpha &= \underline{9.7921–10} \\ \log \sin \beta &= 9.8955–10 \end{aligned} \qquad \begin{aligned} \beta &= \mathbf{51° \; 50'}; & \beta' &= \mathbf{128° \; 10'}. \\ \alpha + \beta &= 90° \;\; 7'; & \alpha + \beta' &= 166° \; 27'. \\ \gamma &= \mathbf{89° \; 53'}; & \gamma' &= \mathbf{13° \; 33'}. \end{aligned}$$

Side c and check.

$$\begin{aligned} \log a &= 2.7798 \\ \log \sin \gamma &= \underline{0.0000} \\ \text{sum} &= 2.7798 \\ \log \sin \alpha &= \underline{9.7291–10} \\ \log c &= 2.9877 \end{aligned} \qquad \begin{aligned} \log b &= 2.8832 \\ \log \sin \gamma &= \underline{0.0000} \\ \text{sum} &= 2.8832 \\ \log \sin \beta &= \underline{9.8955–10} \\ \log c &= 2.9877 \end{aligned}$$

$$c = \mathbf{972.0.}$$

Side c′ and check.

$$\begin{aligned} \log a &= 2.7798 \\ \log \sin \gamma' &= \underline{9.3698–10} \\ \text{sum} &= 2.1496 \\ \log \sin \alpha &= \underline{9.7921–10} \\ \log c' &= 2.3575 \end{aligned} \qquad \begin{aligned} \log b &= 2.8832 \\ \log \sin \gamma' &= \underline{9.3698–10} \\ \text{sum} &= 2.2530 \\ \log \sin \beta' &= \underline{9.8955–10} \\ \log c' &= 2.3575 \end{aligned}$$

$$c' = \mathbf{227.8.}$$

EXERCISES 41

Solve the triangles whose given parts are:

1. $a = 31.1$, $b = 37.4$, $\alpha = 27° \; 18'$.
2. $a = .0878$, $b = .0972$, $\alpha = 65° \; 20'$.
3. $a = 114.3$, $c = 134.6$, $\alpha = 58° \; 6.5'$.
4. $b = 2.72$, $c = 5.56$, $\beta = 29° \; 55'$.
5. $b = 1392$, $c = 3218$, $\gamma = 123° \; 39'$.
6. $a = 482.63$, $c = 550.27$, $\alpha = 57° \; 28.3'$.

88. Case IV. Given the three sides, a, b, c.

The angles may be calculated from either the sine, cosine, or tangent of the half-angles. When all three angles are wanted, it is best to use the tangent. There is no solution when one side equals or exceeds the sum of the other two.

Formulas.

$$s = \tfrac{1}{2}(a + b + c); \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

$$\tan \tfrac{1}{2}\alpha = \frac{r}{s-a}; \quad \tan \tfrac{1}{2}\beta = \frac{r}{s-b}; \quad \tan \tfrac{1}{2}\gamma = \frac{r}{s-c}.$$

Check. $\frac{1}{2}(\alpha + \beta + \gamma) = 90°$; $\alpha + \beta + \gamma = 180°$.

Example.

Given $a = 428.63$, $b = 806.26$, $c = 542.45$.

Graphic solution.

This is shown in the figure. By measuring we find $\alpha = 29°$, $\beta = 112°$, $\gamma = 38°$.

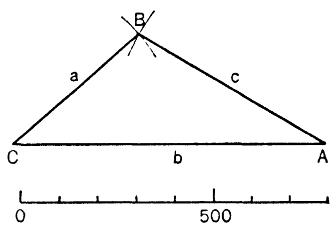


FIG. 67

Logarithmic solution.

Formulas.

$$\log r = \tfrac{1}{2}[\log (s-a) + \log (s-b) + \log (s-c) - \log s].$$

$$\log \tan \tfrac{1}{2}\alpha = \log r - \log (s-a);$$
$$\log \tan \tfrac{1}{2}\beta = \log r - \log (s-b);$$
$$\log \tan \tfrac{1}{2}\gamma = \log r - \log (s-c).$$

Check. $\frac{1}{2}(\alpha + \beta + \gamma) = 90°$. $\alpha + \beta + \gamma = 180°$.

The detailed calculations follow. Five-place tables are used.

$a =$	428.63	$\log (s-a) =$	2.66280	$\frac{1}{2}\alpha =$	14° 47.9′
$b =$	806.26	$\log (s-b) =$	1.91598	$\frac{1}{2}\beta =$	55° 51.6′
$c =$	542.45	$\log (s-c) =$	2.53935	$\frac{1}{2}\gamma =$	19° 20.5′
$2s =$	1777.34	sum $=$	7.11813	*Check.*	90° 00.0′
$s =$	888.67	$\log s =$	2.94875		
$s - a =$	460.04	diff. $=$	4.16938	$\alpha =$	29° 35.8′
$s - b =$	82.41	$\log r =$	2.08469	$\beta =$	111° 43.2′
$s - c =$	346.22	$\log \tan \frac{1}{2}\alpha =$	9.42189 − 10	$\gamma =$	38° 41.0′
Check.	1777.34	$\log \tan \frac{1}{2}\beta =$	0.16871	*Check.*	180° 00.0′
		$\log \tan \frac{1}{2}\gamma =$	9.54534 − 10		

NOTE. The four numbers s, $s - a$, $s - b$, $s - c$ add up to $4s - (a + b + c) = 4s - 2s = 2s$. This checks the numerical work at this stage.

Students who wish to use the cologarithm may write

$$\log r = \tfrac{1}{2}[\log (s - a) + \log (s - b) + \log (s - c) + \operatorname{colog} s].$$

This makes the computation a little more compact.

EXERCISES 42

Solve the triangles whose given parts are:

1. $a = 112$, $b = 86$, $c = 98$.
2. $a = .6852$, $b = .6284$, $c = .6066$.
3. $a = 55.33$, $b = 30.33$, $c = 39.30$.
4. $a = .00150$, $b = .00181$, $c = .00294$.
5. $a = 1626$, $b = 1448$, $c = 3075$.
6. $a = 3.2265$, $b = 2.0842$, $c = 1.8187$.

89. Areas of oblique plane triangles.

Referring to the figures of §78, we see that h is the altitude drawn on side c as base. Hence if K denotes the area of the triangle, we have

$$K = \tfrac{1}{2}hc = \tfrac{1}{2}ac \sin \beta. \quad (h = a \sin \beta.) \tag{8}$$

Hence, *the area of a plane triangle equals half the product of two sides by the sine of their included angle.*

The area is also expressible in simple form in terms of the sides. In the formula above replace $\sin \beta$ by $2 \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta$. Then

$$\begin{aligned} K &= ac \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\beta \\ &= ac\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{s(s-b)}{ac}}, \end{aligned}$$

by (4′) and (5′) of §82. Hence,

$$K = \sqrt{s(s-a)(s-b)(s-c)} = rs. \tag{9}$$

When the given parts of the triangle are such that neither of the above formulas applies directly, it is usually best to calculate additional parts so that one of these formulas may be used.

90. EXERCISES AND PROBLEMS 43

1. $b = 5818$, $\alpha = 36° 56'$, $\beta = 72° 6'$.

2. $a = 91.95$, $b = 29.25$, $c = 83.30$.

3. $c = 1307$, $\alpha = 81° 52'$, $\gamma = 55° 41'$.

4. $b = 167.10$, $\alpha = 65° 49.8'$, $\beta = 38° 37.4'$.

5. $a = 0.2018$, $b = 0.1466$, $\gamma = 58° 47'$.

6. $b = 1032$, $c = 1368$, $\alpha = 23° 7'$.

7. $a = 76.15$, $b = 94.05$, $\alpha = 21° 21'$.

8. $a = 2748$, $b = 8966$, $\alpha = 148° 35'$.

9. $a = 0.04353$, $b = 0.00458$, $c = 0.03951$.

10. $b = 8310$, $c = 6366$, $\gamma = 49° 59.7'$.

11. $a = 2.152$, $c = 1.589$, $\alpha = 19° 12.7'$.

12. $a = 1064$, $b = 1408$, $\gamma = 73°$.

13. $a = 0.1968$, $c = 0.1183$, $\gamma = 22° 32'$.

14. $a = 3828$, $b = 4146$, $c = 2964$.

15. $b = 0.00279$, $c = 0.00233$, $\alpha = 57° 53'$.

16. $a = 2914$, $c = 946$, $\beta = 13° 11.7'$.

17. $a = 0.000598$, $c = 0.000360$, $\alpha = 63° 50'$.

18. $b = 7265$, $c = 3218$, $\gamma = 48° 32'$.

19. $b = 0.5064$, $c = 0.7458$, $\gamma = 10° 32.8'$.

20. $a = 40.369$, $b = 37.403$, $c = 38.088$.

21. $b = 64082$, $\alpha = 13° 31'$, $\beta = 15° 9.4'$.

22. $b = 3236$, $c = 3610$, $\gamma = 56° 34.5'$.

23. $a = 0.01566$, $c = 0.01307$, $\beta = 42° 27'$.

24. $a = 3459$, $\beta = 44° 03'$, $\gamma = 67° 10'$.

25. $b = 0.1974$, $\beta = 51° 41.8'$, $\gamma = 93° 46.1'$.

26. $a = 0.0157$, $b = 0.0428$, $c = 0.0588$.

27. $a = 385.2$, $b = 455.3$, $\alpha = 41° 13'$.

28. $a = 165$, $b = 345$, $\alpha = 69° 18'$.

29. $a = 632$, $b = 741$, $\alpha = 27° 18'$.

30. $a = 10.33$, $b = 5.03$, $c = 6.68$.

31. $b = 3110$, $c = 1466$, $\alpha = 52° 11.2'$.

32. $a = 15.633$, $b = 17.826$, $c = 43.785$.

33. $a = 11782$, $b = 14216$, $\beta = 50° 20.9'$.

34. $a = 44.44$, $b = 77.78$, $\gamma = 58° 49'$.

35. $c = 0.03765$, $\alpha = 45° 29.5'$, $\gamma = 120° 15'$.

36. $a = 10728$, $c = 7574$, $\beta = 104° 20'$.

37. $b = 97.16$, $\alpha = 21° 13.9'$, $\beta = 126° 26.4'$.

38. $a = 675$, $c = 375$, $\alpha = 100° 56.7'$.

39. $a = 0.00932$, $b = 0.00850$, $\beta = 63° 40'$.

40. $a = 0.0762$, $b = 0.0761$, $\beta = 91° 30'$.

In any triangle ABC, whose sides, opposite angles α, β, γ, respectively, are a, b, c, show that:

41. $b(s-b)\cos^2\frac{\alpha}{2} = a(s-a)\cos^2\frac{\beta}{2}$.

42. $a = b\cos\gamma + c\cos\beta$.

43. $(a-b)(1+\cos\gamma) = c(\cos\beta - \cos\alpha)$.

44. $\frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c} = \frac{a^2+b^2+c^2}{2abc}$.

45. $(b+c-a)\tan\frac{\alpha}{2} = (c+a-b)\tan\frac{\beta}{2}$.

46. $(b+c)(1-\cos\alpha) = a(\cos\beta+\cos\gamma)$.

47. $(a^2-b^2+c^2)\tan\beta = (a^2+b^2-c^2)\tan\gamma$.

48. $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$.

49. The radius of the inscribed circle is $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

50. The diameter of the circumscribed circle is $a\csc\alpha$.

51. Find the lengths of diagonals and the area of a parallelogram two of whose sides are 5 ft. and 8 ft., their included angle being 60°.

52. Two adjacent sides of a parallelogram are a and b, their included angle γ; show that the area is $ab\sin\gamma$.

53. The sides of a triangle are in the ratio of 2 : 3 : 4; find the cosine of the smallest angle.

54. The angles of a triangle are as 1 : 2 : 3; the longest side is 100 ft.; solve the triangle.

55. The angles of a triangle are as 3 : 4 : 5; the shortest side is 500 ft.; solve the triangle.

56. The sides of a triangle are 4527, 7861, 6448; find the length of the median drawn to the shortest side. *Ans.* 6824.

57. In $\triangle ABC$, $a = 466$, $b = 572$, $c = 321$. Calculate the shortest altitude. *Ans.* 261.5.

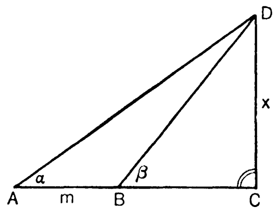
58. In $\triangle ABC$, $a = 336$, $b = 215$, $c = 252$. Calculate the length of the shortest median.

Exercises 59–90, which follow, are problems in "Heights and Distances," so-called; they indicate some of the applications of Trigonometry to mensuration.

For example, the figure of Exercise 59 is a general figure applying to such problems as are illustrated by Exercises 71 and 72 below. The figure of Exercise 70 applies to such problems as appear in Exercises 82 and 83, which represent actual observations of the flight of an airplane and of a meteor respectively.

In the figures, x, the unknown, is to be expressed in terms of the other parts, which are regarded as being given by measurement. Right angles are indicated by a double arc. In each case assume a set of numerical values for the given parts and calculate the numerical value of x.

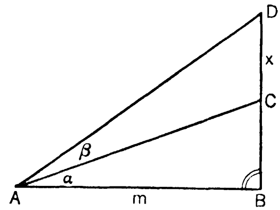
59.



$$x = m\frac{\sin\alpha \sin\beta}{\sin(\beta - \alpha)}.$$

See (§47).

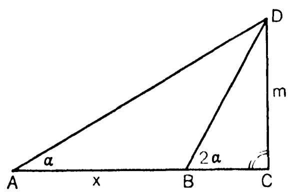
60.



$$x = m\frac{\sin\beta \sec\alpha}{\cos(\alpha + \beta)}$$

($AC = m \sec\alpha$; $\angle ADC = 90° - (\alpha + \beta)$; then apply law of sines to $\triangle ACD$. Or, take $x = BD - BC$.)

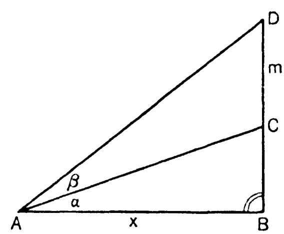
61.



$$x = m \csc 2\alpha.$$

(Note that $\triangle ABD$ is isosceles.)

62.



$$x = m \cos\alpha \csc\beta \cos(\alpha + \beta).$$

(First find AC in $\triangle ACD$.)

63. $x = BC + CD$,

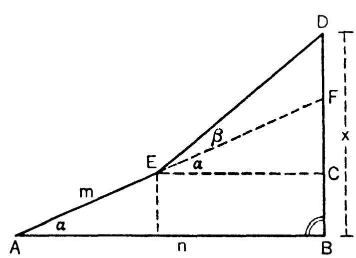
$BC = m \sin\alpha$,

$CD = (n - m\cos\alpha)\tan(\alpha + \beta)$;

or, $x = BF + FD$,

$BF = n \tan\alpha$,

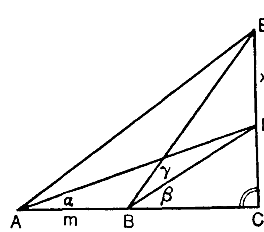
$$FD = (n \sec\alpha - m)\frac{\sin\beta}{\cos(\alpha + \beta)}.$$



64. $$x = m\frac{\sin\alpha \sin\beta}{\sin(\beta - \alpha)}\left[\frac{\tan(\beta + \gamma)}{\tan\beta} - 1\right]$$

$$= m\frac{\sin\alpha \sin\gamma}{\sin(\beta - \alpha)\cos(\beta + \gamma)}.$$

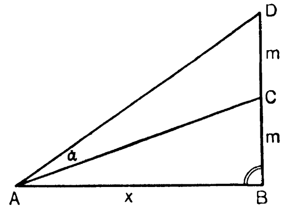
(First find CD as in Ex. 59; then BC, then CE; then $x = CE - CD$. This gives first form; reduce to second form.)



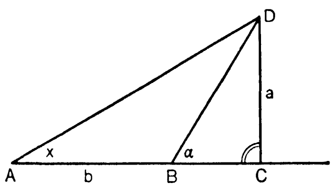
65. $x = \frac{m}{2}\left[\cot \alpha \pm \sqrt{\cot^2 \alpha - 8}\right]$.

(Two solutions).

(Let $\angle BAC = \beta$; then $\tan \beta = \frac{m}{x}$, and $\tan(\alpha + \beta) = \frac{2m}{x}$; expand $\tan(\alpha + \beta)$; substitute value of $\tan \beta$, and solve for x.)



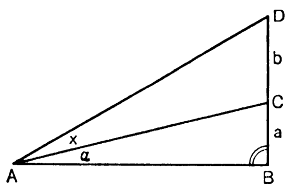
66.



$x = \cot^{-1}\left(\frac{b}{a} + \cot \alpha\right)$.

($\cot x = (AB + BC) \div a$.)

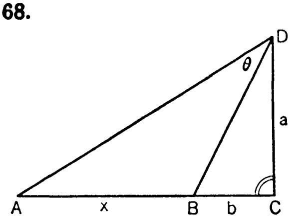
67.



$x = \tan^{-1}\left(\frac{a + b}{a \cot \alpha}\right) - \alpha$.

(Note that $\tan(x + \alpha) = (a + b) \div AB$.)

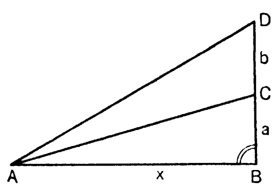
68.



$x = a \cot\left(\tan^{-1}\frac{a}{b} - \theta\right) - b$.

($x + b = a \cot A$; $A = \tan^{-1}\frac{a}{b} - \theta$.)

69.



Given $\angle BAC = \angle CAD$.

$$x = a\sqrt{\frac{b + a}{b - a}}.$$

70.

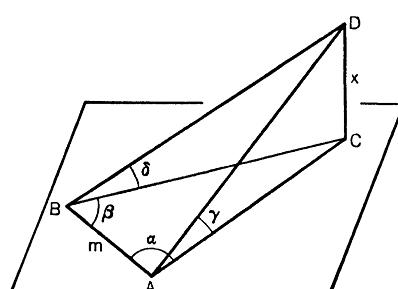
CD is $\perp$ to plane of $\triangle ABC$;

α and β are $\measuredangle$s of $\triangle ABC$;

γ and δ are $\measuredangle$s in vertical planes.

$$x = m\frac{\sin \beta \tan \gamma}{\sin(\alpha + \beta)};$$

or $$x = m\frac{\sin \alpha \tan \delta}{\sin(\alpha + \beta)}.$$



71. From a level plain, the angle of elevation of a distant mountain top is $5° 50'$; after approaching 4 miles, the angle is $8° 40'$; how high is the mountain?

72. From a point on level ground the angle of elevation of the top of a hill is $14° 12'$; on approaching 1000 ft., the angle is $17° 50'$; how high is the hill? *Ans.* 1186 ft.

73. From level ground the angle of elevation of the top of a hill is $11° 30'$; after approaching 3000 ft. up an incline of $3° 27'$, the angle of elevation of the top is $21° 32'$; how high is the hill?

74. From a point 60 ft. above sea level the angle between a distant ship and the sea horizon (the offing) is $20'$; how far away is the ship? (Consider the surface of the sea as a plane, and the distance to the horizon 10 miles.) *Ans.* 8640 ft.

75. A tower 100 feet high has a mark 40 feet above the ground. How far from the foot of the tower will the two parts subtend equal angles?

76. A column 12 feet high stands on a pedestal 8 feet high. How far from the foot of the pedestal (and in the same horizontal plane with it) will column and pedestal subtend equal angles?

77. A flag pole 30 feet high, standing on ground which slopes upward at an angle of 20°, casts a shadow 50 feet long and extending directly down the hill. What is the altitude of the sun?

78. The angle of elevation of the top of a building 100 ft. high is 60°; what will be the angle at double the distance?

79. From a station on level ground due south of a hill, the angle of elevation of the top is 15°; from a point 2000 ft. east of this station the angle of elevation is 12°; how high is the hill?

80. On level ground, 250 ft. from the foot of a building, the angles of elevation of the top and bottom of a flag pole surmounting the building are $38° 43'$ and $31° 2'$ respectively; find the height of the building and the pole.

81. A flag pole on a building subtends an angle of $7° 40'$ at a point on the ground 100 ft. from the building; on approaching 20 ft., the pole subtends an angle of $7° 50'$; find the height of the pole and the building.

82. To determine the height of an airplane, simultaneous observations from two stations were made as follows (see Ex. 70): $m = 6236$ ft.; $\alpha = 72° 12'$, $\beta = 74° 10'$, $\gamma = 9° 24'$, $\delta = 9° 37'$. Show that the average of the two values of h is 1803 ft.

83. To determine the height of a meteor, simultaneous observations from two stations were made as follows (see Ex. 70): $m = 18.3$ miles; $\alpha = 56° 35'$, $\beta = 104° 30'$, $\gamma = 53° 50'$, $\delta = 56° 45'$. Show that the average of the two values of h is 72.0 miles.

84. On approaching 1 mile toward a hill, the angle of elevation of its top is doubled; on approaching $\frac{2}{3}$ mile more, the angle is again doubled; how high is the hill? *Ans.* $\frac{1}{4}\sqrt{7}$ mi.

85. A building surmounted by a flag pole 20 ft. high stands on level ground. From a point on the ground the angles of elevation of the top and the bottom of the pole are $53° 5'$ and $45° 11'$ respectively. How high is the building?

86. A and B are two points neither of which is visible from the other. To determine the distance AB, two stations C and D are chosen and the following measurements made: $CD = 500.0$ ft.; $\angle ACD = 30° 25' 15''$; $\angle ACB = 85° 40' 20''$; $\angle BDC = 35° 14' 50''$; $\angle BDA = 80° 20' 25''$; find AB. *Ans.* 969.2 ft.

87. In a chain of three non-overlapping triangles, the following data are known:

$$AB = 1000 \text{ ft.}$$

$\triangle ABC$,	$\triangle ACD$,	$\triangle CDE$,
$\angle A = 44° 36'$,	$\angle A = 56° 32'$,	$\angle C = 55° 30'$,
$\angle C = 40° 0'$;	$\angle C = 50° 20'$;	$\angle E = 77° 02'$;

calculate DE. (Express DE in terms of AB and the necessary angles by the law of sines.)

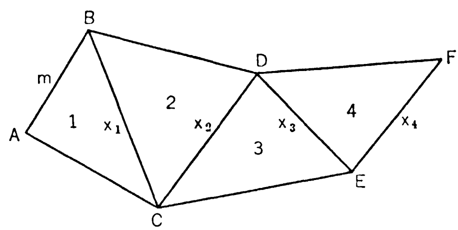
88. In a chain of four non-overlapping triangles, the following data are known:

$$AB = 11{,}289 \text{ meters.}$$

$\triangle ABC$,	$\triangle CBD$,	$\triangle DBE$,	$\triangle DEF$,
$\angle A = 58° 10' 35''$,	$\angle B = 86° 50' 0''$,	$\angle D = 79° 12' 8''$,	$\angle D = 50° 41' 5''$,
$\angle B = 69° 55' 0''$;	$\angle C = 46° 48' 0''$;	$\angle B = 73° 29' 10''$;	$\angle E = 45° 20' 40''$;

calculate EF. *Ans.* 19955 m.

89. The adjacent figure shows a chain of four triangles in which all the angles, and $AB = m$, are known. To designate the angles we use C_1, C_2, C_3 for the three angles at C, and similarly for the other vertices. Calculate in turn x_1, x_2, x_3, x_4, and show that



$$x_4 = m\frac{\sin A_1 \sin B_2 \sin C_3 \sin D_4}{\sin C_1 \sin D_2 \sin E_3 \sin F_4}.$$

(Exercise 88 gives such a chain of triangles taken from the Transcontinental Triangulation of the U. S. Geodetic Survey.)

90. A tower 50 ft. high stands on the edge of a cliff 150 ft. high. At what distance from the foot of the cliff will the tower subtend an angle of 5°? *Ans.* 59.1 or 513 ft.

91. A right triangle whose perimeter is 100 ft. rests with its hypotenuse on a plane, the vertex of the right angle being 10 ft. from the plane. The angle between the plane of the triangle and the supporting plane is 30°. Find the sides of the triangle.

92. The sides of a triangle are 100, 150, 200 ft. At the vertex of the smallest angle a line 100 ft. long is drawn perpendicular to the plane of the triangle. Find the angles subtended at the farther end of this line by the sides of the triangle.

93. An equilateral triangle 50 ft. on a side rests with one side on a plane with which its plane makes an angle of 60°. How far is the third vertex from the plane?

94. As in exercise 93, if the triangle, instead of being equilateral, has sides 40, 20, 30 ft. and rests on the shortest side. *Ans.* $\frac{45\sqrt{5}}{4}$.

95. The sides of a triangle are as $4:2:3$, and the longest median is 10 ft. Find the sides and angles.

96. The following measurements of a field $ABCD$ are made: A to B, due north, 10 chains; B to C, N. 30° E., 6 chains; C to D, due east, 8 chains; calculate AD, and the area of the field in acres. (1 chain = 4 rods.)
Ans. 18.76 ch.; 7.578 A.

97. The following measurements of a field $ABCDE$ are made: A to B, due east, 25.52 chains; B to C, E. 40° 26′ N., 22.25 chains; C to D, N. 48° 26′ W., 33.75 chains; D to E, W. 31° 15′ S., 18.32 chains; calculate EA and the area of the field in acres.

98. In the field of exercise 96 how much area is cut off by a line due east through B? *Ans.* 3.62 acres, south of dividing line.

99. In the field of exercise 97 where should an east and west line be drawn so as to bisect the area?

100. In the field of exercise 97 where should a north and south line be drawn to cut off 30 acres from the western part of the area?
Ans. 10.892 ch. east of A.

101. If P be the pull required to move a weight W up a plane inclined to the horizontal at an angle i, and μ the coefficient of friction, then

$$P = W\frac{\sin i + \mu \cos i}{\cos i - \mu \sin i}.$$

Calculate P when $W = 1000$ lbs., $i = 30°$, $\mu = 0.1$.

102. In exercise 101, what is i if $P = \frac{1}{2}W$ and $\mu = 0.1$? *Ans.* $\tan^{-1}\frac{8}{19}$.

103. If l be the length of a plane inclined to the horizontal at an angle i, μ the coefficient of friction and g the acceleration due to gravity (32+ ft. per sec. per sec.) the time in seconds required by a body to slide down the plane is

$$T = \sqrt{\frac{2l}{g(\sin i - \mu \cos i)}}.$$

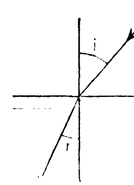
What is T when $l = 25$ ft., $i = 20°$, $\mu = 0.1$?

104. In exercise 103, find i when $l = 100$ ft., $\mu = 0.1$, $T = 5$ sec.
Ans. 20° 7′.

105. When light passes from a rarer to a denser medium, the index of refraction μ is determined by the equation

$$\mu = \frac{\sin i}{\sin r}.$$

When $\mu = 1.2$, what must be i (angle of incidence) to give a deflection of 10°?



106. Find the total deflection of a ray which passes through a wedge whose angle is 30° and index of refraction 1.4, if the ray enters the wedge so that the angle of incidence is 25°, and moves in a plane ⊥ to the edge of the wedge. *Ans.* 12° 32′.

107. Solve exercise 106 when the angle of the wedge is α, the angle of incidence i, and the index of refraction μ.

CHAPTER

IX

INVERSE FUNCTIONS. TRIGONOMETRIC EQUATIONS.

91. Inverse trigonometric functions.

Before proceding with this section the student should review thoroughly §36, where the inverse trigonometric functions and their principal values are defined and illustrated by examples.

Notation.

(a) As in §36, when we write the symbol for an inverse function with the first letter *capitalized*, such as

$$\text{Arc sin } \tfrac{1}{2},\ \text{Arc tan } 1,\ \text{Sec}^{-1}(-2),$$

it shall be understood that the *principal value* is meant.

Thus: $\text{Arc sin } \tfrac{1}{2} = 30°$, $\text{Arc tan } 1 = \dfrac{\pi}{4}$, $\text{Sec}^{-1}(-2) = 120°$.

(b) The non-capitalized form shall indicate the *general value* of an inverse function. So the symbols

$$\text{arc sin } \tfrac{1}{2},\ \text{arc tan } 1,\ \sec^{-1}(-2)$$

mean, in each case, the whole set of angles corresponding to the given function value.

Thus:
$$\text{arc sin } \tfrac{1}{2} = \frac{\pi}{6} + 2n\pi \text{ and } \frac{5\pi}{6} + 2n\pi;$$
$$\text{arc tan } 1 = \frac{\pi}{4} + 2n\pi \text{ and } -\frac{3\pi}{4} + 2n\pi;$$
$$\sec^{-1}(-2) = \frac{2\pi}{3} + 2n\pi \text{ and } -\frac{2\pi}{3} + 2n\pi.$$

(c) When a special notation has not been defined it is necessary to state explicitly in each case whether the general value or the principal value is meant. Thus:

$$\theta = \text{the general value of arc tan } 2;$$
$$\alpha = \text{the principal value of arc tan } 2.$$

Where inverse trigonometric functions are used in other fields of mathematics the reader is often left to decide for himself what meaning to attach to the inverse function symbol.

By use of the definition of principal values the student should check carefully the following statements.

(1) When x is positive, the principal value of each of the six inverse functions,

Arc sin x	Arc tan x	Arc cos x
Arc csc x	Arc cot x	Arc sec x

lies between 0 and $\frac{\pi}{2}$, inclusive of one or both of these values.

(2) When x is negative, the principal value of

Arc sin x	Arc tan x
Arc csc x	Arc cot x

lies between 0 and $-\frac{\pi}{2}$, inclusive of one or both of these values;

the principal value of

Arc cos x
Arc sec x

lies between $\frac{\pi}{2}$ and π, inclusive of one or both of these values.

x negative $\text{Cos}^{-1} x$, $\text{Sec}^{-1} x$ in quad. II	*x positive* p. v. of all six inverse functions in quad. I
	x negative $\text{Sin}^{-1} x$, $\text{Tan}^{-1} x$, $\text{Csc}^{-1} x$, $\text{Cot}^{-1} x$ in quad. IV

These statements are represented schematically in the adjacent diagram.

92. Graphs of the inverse trigonometric functions.

If in the equation $y = \text{arc sin } x$ we solve for x we obtain $x = \sin y$. The two equations are equivalent in that they express exactly the same relation between x and y. Therefore we shall study the graph of the equation $x = \sin y$.

We may start with the equation $y = \sin x$, the fundamental sine wave. Interchanging x with y gives $x = \sin y$, the inverse function equation.

Therefore we obtain the graph of $y = \text{arc sin } x$ by merely interchanging the letters on the coordinate axes in the graph of $y = \sin x$.

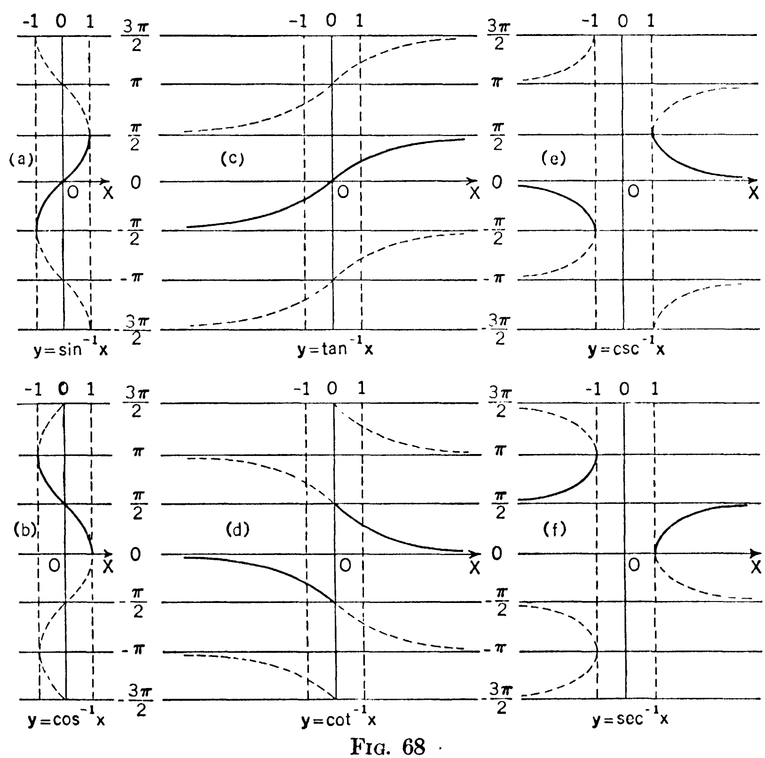


FIG. 68

The graphs of the other inverse functions are related similarly to the corresponding direct functions.

In the figures the principal values are shown by the parts of the curves drawn in the full lines.

93. *Examples.*

1. The principal angle whose sine is $\frac{\sqrt{2}}{2} = ?$

$$\text{Arc sin}\left(\frac{\sqrt{2}}{2}\right) = 45° = \frac{\pi}{4}.$$

2. The principal angle whose sine is $-\frac{\sqrt{2}}{2} = ?$

$$\text{Arc sin}\left(-\frac{\sqrt{2}}{2}\right) = -45° = -\frac{\pi}{4}.$$

3. The principal angle whose secant is $2 = ?$

$$\text{Sec}^{-1}(2) = 60° = \frac{\pi}{3}.$$

4. The principal angle whose secant is $-2 = ?$

$$\text{Sec}^{-1}(-2) = 120° = \frac{2\pi}{3}.$$

5. The principal angle whose cotangent is $-\sqrt{3} = ?$

$$\text{Cot}^{-1}(-\sqrt{3}) = -30° = -\frac{\pi}{6}.$$

6. The principal angle whose cosine is $-0.8382 = ?$

$$\text{Arc cos}(-0.8382) = 180° - 33°\,3' = 146°\,57'.$$

7. $\tan(\text{Arc sin } 0.5) = ?$

We have to find the tangent of the principal angle whose sine is 0.5. That is, if $y = \text{Arc sin } 0.5$, we have to find $\tan y$.

Solution. From $y = \text{Arc sin } 0.5$ we have $\sin y = 0.5 = \frac{1}{2}$. Also, y is in quadrant I. Therefore, taking ordinate $= 1$, distance $= 2$, we obtain abscissa $= \sqrt{3}$. Then

$$\tan y = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}; \quad \tan(\text{Arc sin } 0.5) = \frac{\sqrt{3}}{3}.$$

In this example it happens that $y = 30°$, so that we get immediately $\tan y = \tan 30° = \frac{\sqrt{3}}{3}$. But we can solve the problem without using the value of the angle.

8. tan (arc sin 0.5) = ?

We have to find the tangent of any angle whose sine is 0.5. That is, if $y = \text{arc sin } 0.5$ we have to find $\tan y$.

Solution. Proceeding as before we find abscissa $= \pm \sqrt{3}$.

$$\tan y = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}; \quad \tan(\text{arc sin } 0.5) = \pm \frac{\sqrt{3}}{3}.$$

9. $\tan \text{Arc sin } \frac{2}{3} = ?$

Let $y = \text{Arc sin } \frac{2}{3}$; $\sin y = \frac{2}{3}$; y in quadrant I.

Ordinate = 2, distance = 3; abscissa $= \sqrt{5}$.

$$\tan y = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}; \quad \tan \text{Arc sin } \frac{2}{3} = \frac{2\sqrt{5}}{5}.$$

Obviously we would find $\tan\left(\text{arc sin } \frac{2}{3}\right) = \pm \frac{2\sqrt{5}}{5}$.

10. $\tan \text{Arc sin } (-\frac{2}{3}) = ?$

Let $y = \text{Arc sin } (-\frac{2}{3})$; $\sin y = -\frac{2}{3}$; y in quadrant IV.

Ordinate $= -2$, distance = 3; abscissa $= \sqrt{5}$.

$$\tan y = \tan \text{Arc sin } \left(-\frac{2}{3}\right) = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}.$$

11. $\sec \text{Tan}^{-1} 2 = ?$

Let $y = \text{Tan}^{-1} 2$, or $\tan y = 2$; y in quadrant I.

Take ordinate = 2, abscissa = 1; then distance $= \sqrt{5}$.

$$\sec y = \sec \text{Tan}^{-1} 2 = \sqrt{5}.$$

We might also write

$$\sec y = \sqrt{1 + \tan^2 y} = \sqrt{1 + 4} = \sqrt{5}.$$

12. $\sec (2 \text{ Tan}^{-1} 2) = ?$

We have to find the secant of twice the principal angle whose tangent is 2.

Let $y = \text{Tan}^{-1} 2$; $\tan y = 2$; y in quadrant I.

To find $\sec 2y$ we first obtain $\cos 2y = \cos^2 y - \sin^2 y$.

From $\tan y = 2$ we find $\cos y = \frac{1}{\sqrt{5}}$, $\sin y = \frac{2}{\sqrt{5}}$.

$$\cos 2y = \tfrac{1}{5} - \tfrac{4}{5} = -\tfrac{3}{5}; \quad \sec 2y = -\tfrac{5}{3}.$$

13. $\cos \frac{1}{2}\text{Sec}^{-1}(-3) = ?$

Let $y = \text{Sec}^{-1}(-3)$; $\sec y = -3$; y in quadrant II.

We must find the value of $\cos \frac{1}{2}y$, the angle $\frac{1}{2}y$ being in quadrant I.

$$\cos \tfrac{1}{2}y = +\sqrt{\frac{1+\cos y}{2}} \quad \text{and} \quad \cos y = \frac{1}{\sec y} = -\frac{1}{3}.$$

Therefore $$\cos \tfrac{1}{2}y = \frac{\sqrt{3}}{3} = \cos \tfrac{1}{2}\text{Sec}^{-1}(-3).$$

94. EXERCISES 44

In Exercises 1–20, state the exact value of the principal angle in degrees and radians. Also state the general value of the angle.

1. $\text{arc cos}\,\frac{\sqrt{3}}{2}$.

2. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

3. $\sin^{-1}(-1)$.

4. $\text{arc tan}\,\sqrt{3}$.

5. $\sec^{-1} 2$.

6. $\text{arc cos}\,(-1)$.

7. $\text{arc csc}\,(-2)$.

8. $\cos^{-1}(-\frac{1}{2})$.

9. $\text{arc tan}\,\frac{1}{\sqrt{3}}$.

10. $\text{arc tan}\,(-1)$.

11. $\sin^{-1} 1$.

12. $\sec^{-1} 1$.

13. $\text{arc sec}\left(-\frac{2}{\sqrt{3}}\right)$.

14. $\csc^{-1}\frac{2}{\sqrt{3}}$.

15. $\cot^{-1} 0$.

16. $\text{arc cot}\,(-\sqrt{3})$.

17. $\cot^{-1}\sqrt{3}$.

18. $\csc^{-1} 1$.

19. $\text{arc sin}\left(-\frac{\sqrt{3}}{2}\right)$.

20. $\tan^{-1} 0$.

In Exercises 21–40 obtain the principal angle to the nearest minute.

21. $\text{Arc cos}\,0.2$.

22. $\text{Tan}^{-1}(-3)$.

23. $\text{Sec}^{-1}\sqrt{3}$.

24. $\text{Arc sec}\,4$.

25. $\text{Cos}^{-1}(-0.6)$.

26. $\text{Cos}^{-1}(1-\sqrt{2})$.

27. $\text{Arc tan}\,3$

28. $\text{Sec}^{-1}(-4)$.

29. $\text{Tan}^{-1}(1+\sqrt{2})$.

30. $\text{Arc sin}\,(\frac{1}{3})$.

31. $\text{Cot}^{-1}(-\frac{2}{3})$.

32. $\text{Sin}^{-1}(\sqrt{3}-1)$.

33. $\text{Arc cot}\,(\frac{3}{4})$.

34. $\text{Sin}^{-1}(-\frac{5}{6})$.

35. $\cot^{-1}(2-\sqrt{5})$.

36. $\text{Arc csc}\,2.5$.

37. $\text{Tan}^{-1}(-\frac{1}{3})$.

38. $\text{Csc}^{-1}(1-\sqrt{5})$.

39. $\text{Arc sin}\,0.8$.

40. $\text{Csc}^{-1}(-1.5)$.

In Exercises 41–60 obtain the exact numerical values.

41. $\sin \text{Arc tan}\,3$.

42. $\sin 2\text{Tan}^{-1} 3$.

43. $\cos \frac{1}{2}\sin^{-1} 0.6$.

44. $\tan \text{Arc tan}\,3$.

45. $\sec 2\text{Cot}^{-1} 2$.

46. $\tan \frac{1}{2}\tan^{-1} 1$.

47. $\cos \text{Arc cot}\,2$.

48. $\cos 2\text{Sin}^{-1} 0.8$.

49. $\sin \frac{1}{2}\tan^{-1}\frac{3}{4}$.

50. $\sec \text{Arc sin}\,(\frac{3}{5})$.

51. $\tan 2\text{Sec}^{-1} 1.5$.

52. $\sec \frac{1}{2}\cos^{-1}\frac{5}{13}$.

53. $\tan \text{Arc csc}\,2$.

54. $\cot 2\text{Cos}^{-1} 0.6$.

55. $\cot \frac{1}{2}\tan^{-1}\frac{12}{5}$.

56. $\cos \text{Arc cos}\,0.3$.

57. $\cos 2\text{Cot}^{-1} 0.6$.

58. $\csc \frac{1}{2}\sec^{-1} 2$.

59. $\cot \text{Arc sec}\,1.5$.

60. $\sin 2\text{Sin}^{-1} 0.6$.

95. Equations involving several inverse functions.

Example 1.

Show that Arc sin $\frac{3}{5}$ = Arc cos $\frac{4}{5}$.

Let $\alpha = \text{Arc sin } \frac{3}{5}; \quad \beta = \text{Arc cos } \frac{4}{5}.$

Then $\sin \alpha = \frac{3}{5}; \quad \cos \beta = \frac{4}{5}.$

To prove that $\alpha = \beta,$

or that $\sin \alpha = \sin \beta.$

(The sine function is used for convenience; any other function might be used.)

From $\cos \beta = \frac{4}{5}$ we obtain $\sin \beta = \sqrt{1 - \cos^2 \beta} = \frac{3}{5}$.

Therefore $\sin \alpha = \sin \beta$, and also $\alpha = \beta$, since α and β are both acute angles.

Example 2.

Show that $\text{Tan}^{-1} 2 + \text{Tan}^{-1} 3 = 135°$.

Let $\alpha = \text{Tan}^{-1} 2; \quad \beta = \text{Tan}^{-1} 3.$

Then $\tan \alpha = 2; \quad \tan \beta = 3.$

To prove that $\alpha + \beta = 135°;$

or that $\tan (\alpha + \beta) = \tan 135° = -1.$

Proof. $\tan (\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \dfrac{2 + 3}{1 - 2\cdot 3} = -1.$

Therefore $\alpha + \beta = 135°$, since α and β are positive acute angles and $\tan (\alpha + \beta) = -1$.

Example 3.

Show that $\text{Sin}^{-1} \frac{4}{5} + 2 \text{ Tan}^{-1} 2 = \pi$.

Let $x = \text{Sin}^{-1} \frac{4}{5}; \quad y = \text{Tan}^{-1} 2.$

Then $\sin x = \frac{4}{5}; \quad \tan y = 2.$

To prove that $x + 2y = \pi,$

or that $2y = \pi - x,$

or that $\sin 2y = \sin (\pi - x) = \sin x = \frac{4}{5}.$

From $\tan y = 2$, and the fact that y is a positive acute angle, we find that $\sin y = \dfrac{2}{\sqrt{5}}$ and $\cos y = \dfrac{1}{\sqrt{5}}$.

Then $\sin 2y = 2 \sin y \cos y = \frac{4}{5} = \sin x.$

Example 4.

Show that $\text{Tan}^{-1} \frac{2}{3} + \text{Tan}^{-1} 2 + \text{Tan}^{-1} 8 = \pi$.

Let $x = \text{Tan}^{-1} \frac{2}{3}; \quad y = \text{Tan}^{-1} 2; \quad z = \text{Tan}^{-1} 8;$

then $\tan x = \frac{2}{3}; \quad \tan y = 2; \quad \tan z = 8.$

To prove that $$x + y + z = \pi,$$
or that $$x + y = \pi - z,$$
or that $$\tan(x + y) = \tan(\pi - z) = -\tan z.$$
Now $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{2}{3} + 2}{1 - \frac{4}{3}} = -8 = -\tan z.$$

Example 5.

Show that $\operatorname{Tan}^{-1} a = \operatorname{Sin}^{-1} \dfrac{a}{\sqrt{1 + a^2}}$ when a is positive.

Let $$x = \operatorname{Tan}^{-1} a \quad \text{and} \quad y = \operatorname{Sin}^{-1} \frac{a}{\sqrt{1 + a^2}};$$
then $$\tan x = a \quad \text{and} \quad \sin y = \frac{a}{\sqrt{1 + a^2}}.$$
To prove that $$x = y,$$
or that $$\sin x = \sin y.$$

Now since x and y stand for principal values, and a is positive, both angles are in the first quadrant.

Then from $\tan x = a$ we find
$$\sin x = \frac{a}{\sqrt{1 + a^2}},$$
which is $\sin y$.

96. EXERCISES 45

Verify each of the equations below.

1. $\operatorname{Arc} \tan \frac{12}{5} = \operatorname{Arc} \cos \frac{5}{13}$.
2. $\operatorname{Arc} \sin \dfrac{3}{5} + \operatorname{Arc} \sin \dfrac{4}{5} = \dfrac{\pi}{2}$.
3. $\operatorname{Arc} \sin \frac{3}{5} = \operatorname{Arc} \tan \frac{3}{4}$.
4. $\operatorname{Arc} \tan \frac{4}{5} + \operatorname{Arc} \tan \frac{1}{9} = 45°$.
5. $2 \operatorname{Tan}^{-1} \frac{2}{3} = \operatorname{Tan}^{-1} \frac{12}{5}$.
6. $\operatorname{Tan}^{-1} (-3) = \operatorname{Tan}^{-1} 2 - \dfrac{3\pi}{4}$.
7. $\operatorname{Cot}^{-1} 2 + \operatorname{Csc}^{-1} \sqrt{10} = 45°$.
8. $\operatorname{Sin}^{-1} \dfrac{\sqrt{3}}{2} + 2 \operatorname{Cos}^{-1} \dfrac{\sqrt{3}}{2} = 120°$.
9. $2 \operatorname{Arc} \tan 4 + \operatorname{Arc} \sin \frac{8}{17} = \pi$.
10. $2 \operatorname{Arc} \cot 2 = \operatorname{Arc} \sec \frac{5}{3}$.
11. $3 \operatorname{Sin}^{-1} \frac{1}{4} = \operatorname{Sin}^{-1} \frac{11}{16}$.
12. $4 \operatorname{Tan}^{-1} \dfrac{1}{5} = \operatorname{Tan}^{-1} \dfrac{1}{239} + \dfrac{\pi}{4}$.
13. $\operatorname{Arc} \tan \frac{11}{16} + \operatorname{Arc} \tan \frac{2}{5} + \operatorname{Arc} \tan (-\frac{3}{2}) = \pi$.
14. $\operatorname{Sin}^{-1} \frac{4}{5} + \operatorname{Sin}^{-1} \frac{8}{17} + \operatorname{Sin}^{-1} \frac{13}{85} = \dfrac{\pi}{2}$.
15. $\operatorname{Arc} \cos \frac{63}{65} + 2 \operatorname{Arc} \tan \frac{1}{5} = \operatorname{Arc} \sin \frac{3}{5}$.
16. $2 \operatorname{Tan}^{-1} \frac{2}{3} - \operatorname{Csc}^{-1} \frac{5}{3} = \operatorname{Sin}^{-1} \frac{33}{65}$.
17. $\operatorname{Sin}^{-1} a = \operatorname{Cos}^{-1} \sqrt{1 - a^2}$, if $a > 0$.
18. $2 \operatorname{Tan}^{-1} m = \operatorname{Tan}^{-1} \dfrac{2m}{1 - m^2}$.
19. $2 \operatorname{Tan}^{-1} (\cos 2\theta) = \operatorname{Tan}^{-1} \left(\dfrac{\cot^2 \theta - \tan^2 \theta}{2} \right)$.

NOTE. The equation of Exercise 12 was used to calculate the value of π to 707 places. (See American Mathematical Monthly, vol. 31, page 393, 1924.)

97. Trigonometric equations. Special methods.

In §41 we solved some trigonometric equations, following a *rule* there stated and using the formulas of group A. This section should now be reviewed.

We now have at our disposal all the formulas of the other groups and shall illustrate by some examples how they may be used to solve trigonometric equations.

Example 1.

$$2 \sin^2 x - 3 \sin x \cos x = 1.$$

Since $2 \sin^2 x = 1 - \cos 2x$ and $2 \sin x \cos x = \sin 2x$, we have

$$1 - \cos 2x - \tfrac{3}{2} \sin 2x = 1, \quad \text{or} \quad \tan 2x = -\tfrac{2}{3}.$$

Hence $2x = \tan^{-1}(-\tfrac{2}{3}) = -33° 41' + n\,360°$, or $146° 19' + n\,360°$.

$$x = -16° 50.5' + n\,180°, \text{ or } 73° 9.5' + n\,180°.$$

Exercise. Check these answers. Solve the given equation by expressing $\cos x$ in terms of $\sin x$.

Example 2.

$$\sin 3y - \sin 2y = 0.$$

By formula (24) of §75 this becomes

$$2 \cos \tfrac{5}{2}y \sin \tfrac{1}{2}y = 0.$$

Hence $\cos \tfrac{5}{2}y = 0$ or $\sin \tfrac{1}{2}y = 0$; $\tfrac{5}{2}y = \cos^{-1} 0$, or $\tfrac{1}{2}y = \sin^{-1} 0$.

$y = \tfrac{2}{5} \cos^{-1} 0 = \tfrac{2}{5}(90° + n\,360°)$ or $\tfrac{2}{5}(-90° + n\,360°) = \pm 36° + n\,288°$

$y = 2 \sin^{-1} 0 = 2 \cdot n\pi = n\,360°.$

Example 3.

$$\cos x + \cos 3x + \cos 5x = 0.$$

Since $\cos x + \cos 5x = 2 \cos 3x \cos 2x$,

we have $2 \cos 3x \cos 2x + \cos 3x = 0$, or $\cos 3x(2 \cos 2x + 1) = 0$.

Hence $\cos 3x = 0$, or $\cos 2x = -\tfrac{1}{2}$; $3x = \cos^{-1} 0$, or $2x = \cos^{-1}(-\tfrac{1}{2})$.

$x = \tfrac{1}{3} \cos^{-1} 0 = \tfrac{1}{3}(90° + n\,360°)$ or $\tfrac{1}{3}(-90° + n\,360°) = \pm 30° + n \cdot 120°$

$x = \tfrac{1}{2} \cos^{-1}(-\tfrac{1}{2}) = \tfrac{1}{2}(\pm 120° + n\,360°) = \pm 60° + n\,180°.$

Example 4.

$$\sin 3x = \cos 5x.$$

Change $5x$ to the complementary angle $90° - 5x$:

$$\sin 3x = \sin (90° - 5x); \ \sin 3x - \sin (90° - 5x) = 0.$$

Use formula (24), §75, to change to a product:

$$2\cos\frac{3x+90^\circ-5x}{2}\sin\frac{3x-90^\circ+5x}{2}=0,$$

$$2\cos(45^\circ-x)\sin(4x-45^\circ)=0.$$

Equate each factor to zero:

$$\cos(45^\circ-x)=0,\quad\text{or}\quad\sin(4x-45^\circ)=0.$$

The first factor gives

$$45^\circ-x=\cos^{-1}0=\pm 90^\circ+n\,360^\circ.$$

$$x=-45^\circ-n\,360^\circ\quad\text{or}\quad 135^\circ-n\,360^\circ.$$

(The term $-n\,360^\circ$ may also be written $+n\,360^\circ$, since n stands for any integer, positive or negative.)

The second factor gives

$$4x-45^\circ=\text{arc}\sin 0=n\,180^\circ.$$

$$x=11^\circ\,15'+n\cdot 45^\circ.$$

Check. Both sets of answers check.

NOTE. The equation $\csc 3x=\sec 5x$ may be changed to $\sin 3x=\cos 5x$ by taking reciprocals.

Example 5.

$$\tan 4\theta\tan 5\theta=1.$$

$$\frac{\sin 4\theta}{\cos 4\theta}\frac{\sin 5\theta}{\cos 5\theta}=1;\quad\cos 4\theta\cos 5\theta-\sin 4\theta\sin 5\theta=0.$$

$$\cos(4\theta+5\theta)=\cos 9\theta=0;\quad 9\theta=\pm 90^\circ+n\,360^\circ.$$

$$\theta=\pm 10^\circ+n\cdot 40^\circ.$$

We must rule out any values of θ such that $\cos 4\theta=0$ or $\cos 5\theta=0$, because these occur as divisors in the given equation.

Exercise. Check the answers for several selected values of n.

Example 6.

$$4\sin\theta+3\cos\theta=2.$$

We might reduce to $\sin\theta$ or $\cos\theta$ and proceed according to the rule of §41, Example 4. A method much preferred in practice is as follows.

In place of 4 and 3 introduce two new constants m and M such that

$$\begin{aligned}4&=m\cos M,\\3&=m\sin M;\end{aligned}\qquad\text{whence}\qquad\begin{aligned}m&=\sqrt{4^2+3^2}=5,\\M&=\tan^{-1}\tfrac{3}{4}.\end{aligned}$$

The given equation then becomes

$$5(\sin\theta\cos M+\cos\theta\sin M)=2\quad\text{or}\quad\sin(\theta+M)=\tfrac{2}{5}.$$

$$\theta+M=\sin^{-1}\tfrac{2}{5},\quad\text{or}\quad\theta=\sin^{-1}\tfrac{2}{5}-M.$$

$$\theta=\sin^{-1}\tfrac{2}{5}-\tan^{-1}\tfrac{3}{4}.$$

Exercise. Given $$a \sin \theta + b \cos \theta = c.$$
Show that the general solution is
$$\theta = \sin^{-1} \frac{c}{\sqrt{a^2 + b^2}} - \tan^{-1} \frac{b}{a}.$$
Indicate some values of a, b, c for which there would be no solution.

98. Graphic solutions.

Such solutions, even when they are only rough approximations, are often very useful. Moreover, an approximate value may be corrected by successive trials to any desired degree of accuracy.

Example 1.

Solve graphically: $\sin 2\theta + \sin \theta + \frac{1}{2} = 0$.

We want the values of θ which reduce the expression $\sin 2\theta + \sin \theta + \frac{1}{2}$ to zero.

Let $$y = \sin 2\theta + \sin \theta + \tfrac{1}{2}.$$

Calculate y for a series of values of θ, as $\theta = 0°, 10°, 20°, \ldots$, and plot the points (θ, y) in rectangular coordinates. The resulting curve

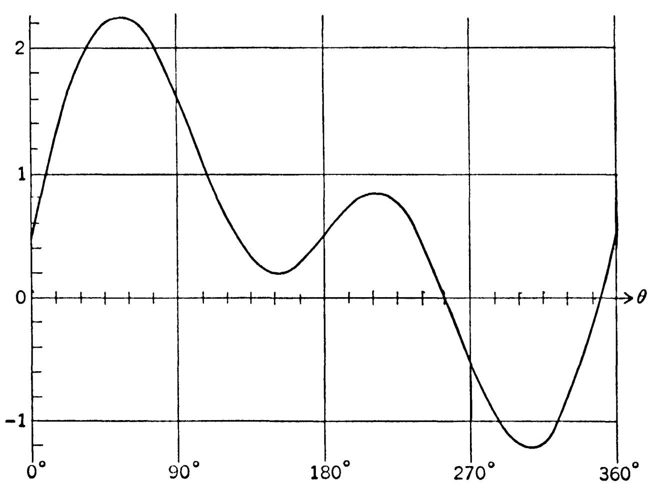


FIG. 69

will show the approximate values of θ for which y is zero. Any convenient scales may be used on the axes of θ and y.

Let the student read off the required solutions from the graph.

NOTE. If the number $\frac{1}{2}$ in the given equation is changed, let us say, to $1\frac{1}{2}$, the effect on the graph will be to raise the entire curve one unit; the same effect could be produced by lowering the angle scale one unit.

EXERCISE

By means of this graph solve the equations

(a) $\sin 2\theta + \sin \theta + 1.5 = 0;$
(b) $\sin 2\theta + \sin \theta = 0;$
(c) $\sin 2\theta + \sin \theta = 1;$
(d) $\sin 2\theta + \sin \theta = \frac{1}{4}.$

Example 2.

Solve graphically: $\tan x = \frac{1}{2}x$, (x in radians).

(a) Draw the graph of
$$y = \tan x.$$

(b) Draw the graph of $y = \frac{1}{2}x$.

(c) Note the points where these graphs intersect. The values of x at these points are the required solutions. The figure indicates $x = 0$ and $x = \pm 4.3$ radians.

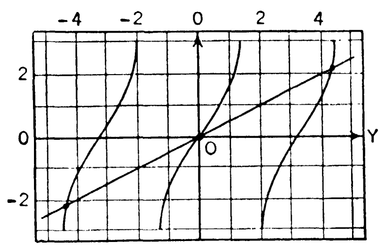


FIG. 70

Example 3.

Solve graphically: $E - 0.9 \sin E = \frac{\pi}{3}.$

This is an example of " Kepler's Equation," a basic equation in the calculation of the position of a planet in its orbit. Angle E is assumed to be in radian measure.

We may solve the equation for $\sin E$:

$$\sin E = \frac{E - \frac{\pi}{3}}{0.9}.$$

(a) Draw the graph of $y = \sin E$.

(b) Draw the graph of $y = \dfrac{E - \frac{\pi}{3}}{0.9}.$

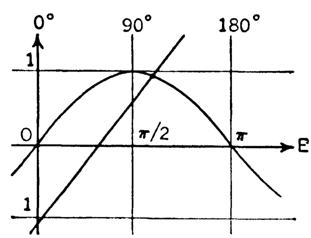


FIG. 71

The first graph is the fundamental sine wave; the second is a straight line. This line was obtained by locating two points on it by use of equation (b) which gives $y = -1.15$ when $E = 0$ and $y = 2.30$ when $E = \pi$. The second point is outside of the bounds of the figure.

The graphs have but one point in common at which we might estimate the value of E as about 112°.

A graphic solution may be regarded as a trial value and corrected by use of the tables. We illustrate by correcting the value of E just found. We compare the value of $0.9 \sin E$ with that of $E - \pi/3$, and change E to make them more nearly equal.

E	$E - \frac{\pi}{3}$		$\sin E$	$0.9 \sin E$	Diff.
112°	52° =	0.908 rad.	0.927	0.834	+ 0.074
110	50	0.873	0.940	0.846	+ 0.027
108	48	0.838	0.951	0.856	− 0.018
109	49	0.855	0.945	0.850	+ 0.005
108° 50′		0.852	0.947	0.852	0.000

The new value of E is 108° 50′. This could be further corrected by use of more extensive tables.

EXERCISES

Solve graphically. Check and correct by use of tables.

1. $3 \tan x = 2x$.

2. $2 \sin x = x$.

3. $3 \sin x = 2x$.

4. $3 \cos x = 2x$.

5. $0.8 \sin x = x - \pi/3$.

6. $0.5 \sin x = x - 30°$.

CHAPTER

X

ANALYTICAL TRIGONOMETRY

99. Polar coordinates.

We have made repeated use of the system of rectangular coordinates, in which the position of any point in the plane is defined by its abscissa and ordinate. A second system of coordinates defines the position of a point with reference to a single fixed line, called the *initial line,* and a fixed point on this line, called the *origin* or *pole.*

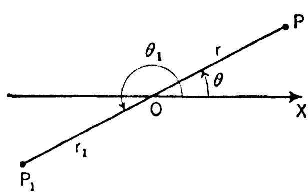


FIG. 72

In the figure, let OX be the initial line and O the pole. We shall consider OX as the positive direction of the initial line. Let P be a point in the plane. The position of P is then fixed by its distance $OP = r$ from O, called the *radius vector,* and by the angle $XOP = \theta$, called the *vectorial angle.* Then r, θ are called the *polar coordinates* of P, and the point is indicated by (r, θ). Similarly P_1 is the point (r_1, θ_1). The coordinate θ is positive when measured counter-clockwise from OX; r is positive when measured from O along the terminal side of θ; it is negative when measured from O along the terminal side of θ produced back through O. Thus the points $(5, 30°)$ and $(-5, 210°)$ coincide. Similarly with $(-3, 135°)$ and $(3, -45°)$.

100. Relation between polar and rectangular coordinates.

Let O be the origin and OX the initial line of a system of polar coordinates (figure). Let OX and OY be the axes of a rectangular system of coordinates. Then

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2},$$
$$y = r\sin\theta; \qquad \theta = \tan^{-1}\frac{y}{x}.$$

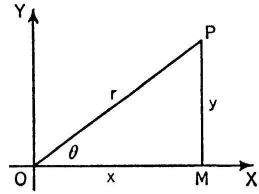


FIG. 73

EXERCISES

Plot the following points:

$(1, 45°)$; $(-1, 45°)$; $(3, 60°)$; $(3, -60°)$; $\left(4, \frac{\pi}{8}\right)$; $\left(2, -\frac{2\pi}{5}\right)$; $\left(\frac{2}{3}, \frac{5\pi}{6}\right)$;
$\left(-\frac{2}{3}, -\frac{5\pi}{6}\right)$; $\left(1, \frac{3\pi}{2}\right)$; $\left(-1, -\frac{3\pi}{2}\right)$; $(\pi, 800°)$.

Calculate the rectangular coordinates of each of these points, taking O as origin and OX as the x-axis.

101. Curves in polar coordinates.

When r and θ are unrestricted, the point (r, θ) may take any position in the plane. When r and θ are connected by an equation, the point (r, θ) is in general restricted to a curve, the equation between r and θ being called the polar equation of the curve.

Example 1.

Trace the curve whose polar equation is $r = \sin\theta$.

Assume a series of values for θ, calculate the corresponding values of r and plot the points whose coordinates are corresponding values of r and θ.

$\theta = 0°, 30°, 60°, 90°, 120°, 150°, 180°,$
$r = 0, 0.5, 0.87, 1.0, 0.87, 0.5, 0,$
$\theta = 210°, 240°, 270°, 300°, 330°, 360°.$
$r = -0.5, -0.87, -1.0, -0.87, -0.5, 0.$

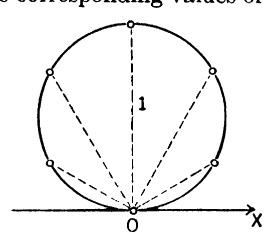


FIG. 74a

The graph is shown in the figure. For values of $\theta > 360°$, and for negative angles, no new points are obtained. The curve is a circle, with radius $= \frac{1}{2}$.

Example 2.

Trace the curve $r = 2\theta$.

Here θ is understood to be in radians.

$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \ldots, 2\pi.$$

$$r = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots, 4\pi.$$

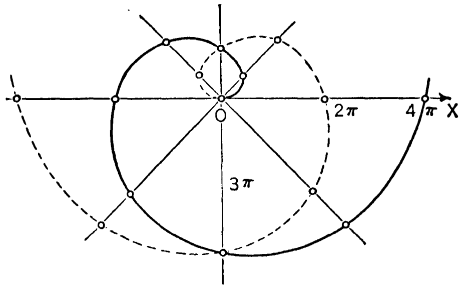


FIG. 74b

For negative values of θ we get corresponding negative values of r. The curve is the double spiral in the figure, the branches shown by the full line and the dotted line being obtained from the positive and the negative values of θ respectively.

EXERCISES

Trace the following curves:

1. $r = 2 \sin \theta$.
2. $r = \cos \theta$.
3. $r = \tan \theta$.
4. $r = \sec \theta$
5. $r = 1 + \cos \theta$.
6. $r = 2 + \sin \theta$.
7. $r\theta = 1$.
8. $r = 2^\theta$.
9. $r = \cos^2 \theta$.
10. $r = \cos 2\theta$.
11. $r = 4$.
12. $\theta = \pi/4$

102. Complex numbers.

Let a and b denote any two *real* numbers and $i = \sqrt{-1}$. More precisely, i is defined by the equation $i^2 = -1$. Then the quantity $a + ib$ is called a *complex number*. It may be considered as made up of a real units and b imaginary units, $a \times 1 + b \times i$.

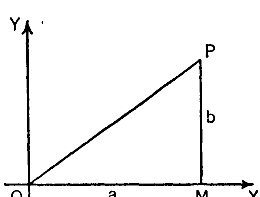


FIG. 75

Real numbers can be represented by points on a straight line. To represent complex numbers geometrically, we require a plane.

Let OX and OY be a system of rectangular axes, and P a point in their plane having coordinates (a, b) (figure). Then the vector OP is considered to represent the complex number $a + ib$, and the extremity of this vector, P, is called the representative point of the complex number $a + ib$.

When $b = 0$, P lies on the x-axis, and the complex number reduces to a real number. Thus all points on the x-axis correspond to real numbers, and this line is called the axis of real numbers.

Let P (figure) be a point (x, y) in the plane, and let z be the complex number represented by P. Then

$$z = x + iy.$$

Now take OX as the initial line and O the pole of a system of polar coordinates. Let the polar coordinates of P be (r, θ).

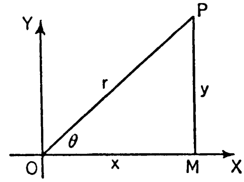


FIG. 76

Then

$$x = r \cos \theta; \qquad y = r \sin \theta.$$

Hence

$$z = x + iy = r (\cos \theta + i \sin \theta).$$

Here r is called the *modulus* and θ the *angle* of the complex number z.

When r is fixed, and θ is changed by integral multiples of 2π, we obtain a set of complex numbers of the form,

$$z = r\left[\cos (\theta + 2n\pi) + i \sin (\theta + 2n\pi)\right];$$
$$n = 0, \quad \pm 1, \quad \pm 2, \ldots .$$

All these numbers have the same representative point.

103. Addition of complex numbers.

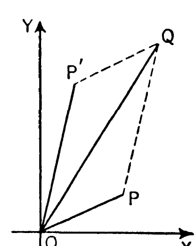


FIG. 77

The sum of two complex numbers,

$$z = x + iy \quad \text{and} \quad z' = x' + iy',$$

is *defined* by the equation

$$z + z' = (x + x') + i(y + y').$$

We proceed to consider this sum geometrically. Let P, P' (figure) be the representative points of z, z' respectively. On OP and OP' as adjacent sides construct the parallelogram $OPQP'$. Then Q is the representative point of $z + z'$. For the

coordinates of Q are $(x + x', y + y')$. This amounts precisely to vector addition of the vectors OP and OP', §52.

The difference of the two complex numbers z and z' may be defined by the equation

$$z - z' = (x - x') + i(y - y').$$

Exercise. Give a geometric construction for the representative point of $z - z'$.

104. Multiplication of complex numbers.

The product of the two complex numbers,

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad z' = r'(\cos\theta' + i\sin\theta'),$$

is *defined* by the equation

$$zz' = rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta'),$$

the binomials to be multiplied in the usual way; thus:

$$\begin{aligned} zz' &= rr'[\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')] \\ &= rr'[\cos(\theta + \theta') + i\sin(\theta + \theta')]. \end{aligned}$$

Therefore the modulus of the product zz' equals the product of the moduli of z and z', and the angle of zz' equals the sum of the angles of z and z'.

By repeating this process we find

$$zz'z'' = rr'r''[\cos(\theta + \theta' + \theta'') + i\sin(\theta + \theta' + \theta'')]$$

and so on, for any finite number of factors.

When the factors are all equal this reduces to

$$z^n = r^n(\cos n\theta + i\sin n\theta),$$

n being a positive integer.

Exercise. Show that the above definition of the product zz' is the same as

$$zz' = xx' - yy' + i(xy' + x'y),$$

where $\quad z = x + iy \quad \text{and} \quad z' = x' + iy'.$

105. De Moivre's theorem.

When $r = 1$, then $z = \cos\theta + i\sin\theta$. Hence by the above result we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This equation contains what is known as *De Moivre's Theorem.*

106. Definition of z^p.

Let p be any *real* number, positive or negative, rational or irrational. Then by analogy with the result for z^n when n is a positive integer, we *define* z^p by the equation

$$z^p = r^p (\cos p\theta + i \sin p\theta),$$

where

$$z = r (\cos \theta + i \sin \theta).$$

Then, if q also be real, we have

$$z^q = r^q (\cos q\theta + i \sin q\theta),$$

and

$$z^p z^q = r^{p+q}[\cos (p + q)\theta + i \sin (p + q)\theta] = z^{p+q}.$$

All the rules for exponents will be the same when the base is a complex number as when the base is real.

Examples.

1. Find the modulus and angle of $z = 3 - 4i$.

Here $\quad 3 = r \cos \theta; \; -4 = r \sin \theta.$

$$\therefore \quad r = \sqrt{3^2 + 4^2} = 5, \tan \theta = \frac{-4}{3},$$

or, $\quad \theta = \tan^{-1}(-\frac{4}{3}).$

The angle lies in the fourth quadrant.

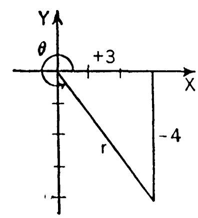


FIG. 78

2. Express $2(\cos 150° - i \sin 150°)$ in the form $x + iy$.

$$2(\cos 150° - i \sin 150°) = 2\left(-\frac{1}{2}\sqrt{3} - \frac{i}{2}\right) = -\sqrt{3} - i.$$

3. Find the value of $(1 + i)^2(2 - 3i)$.

$$(1 + i)^2 = 1 + 2i + i^2 = 2i.$$
$$(1 + i)^2(2 - 3i) = 2i(2 - 3i) = 4i - 6i^2 = 6 + 4i.$$

EXERCISES 46

1. Find the modulus and angle of

$$1 - i; \quad 4 + 3i; \quad -5 + 11i; \quad 2i; \quad 2; \quad (1 + i)(1 - i);$$
$$3\sqrt{3} + 3i; \quad (3\sqrt{3} - 3i)^2; \quad (1 + i\sqrt{3})(\sqrt{3} + i).$$

Give figure for each case.

2. Find the value of:

$$(1 + i)^3; \; (1 - i)^4; \; (1 + i)^2(1 + 2i)^2; \; (3 - 3i)^2(\sqrt{3} + i)^3; \; (1 - i\sqrt{3})^6.$$

107. Theorem.

If P and Q are any real quantities and if $P + iQ = 0$, then $P = 0$ and $Q = 0$.

Proof. By hypothesis, $P + iQ = 0$ or $P = -iQ$.

Squaring, $$P^2 = -Q^2.$$

Now P^2 and Q^2 (if not zero) must be positive, hence the last equation states that a positive quantity equals a negative quantity. This is impossible unless both quantities are zero.

$$\therefore \quad P = 0 \quad \text{and} \quad Q = 0.$$

This theorem is used to replace a given equation of the form

$$P + iQ = 0$$

by the equivalent equations

$$P = 0; \; Q = 0.$$

As a *corollary* we have, if

$$P + iQ = P' + iQ',$$

then $$P = P' \quad \text{and} \quad Q = Q'.$$

For the given equation is equivalent to

$$(P - P') + i(Q - Q') = 0.$$

108. The nth roots of unity.

To solve the equation

$$x^n - 1 = 0, \quad \text{or} \quad x^n = 1,$$

replace 1 by its value $\cos 2k\pi + i \sin 2k\pi$, k being an integer. We obtain

$$x^n = \cos 2k\pi + i \sin 2k\pi.$$

Taking the nth roots of both members we have, by putting $p = \frac{1}{n}$ in §106,

$$x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

Here k may be any integer; letting $k = 0, 1, 2, \cdots n - 1$, we obtain n distinct values of x, that is, n distinct nth roots of 1. For other values of k we obtain the same roots over again.

Geometric representation of the nth roots of unity.

The nth roots of 1 are,

$$k = 0; \qquad x_1 = \cos 0 + i \sin 0 = 1,$$

$$k = 1; \qquad x_2 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},$$

$$k = 2; \qquad x_3 = \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$k = n - 1; \quad x_n = \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}.$$

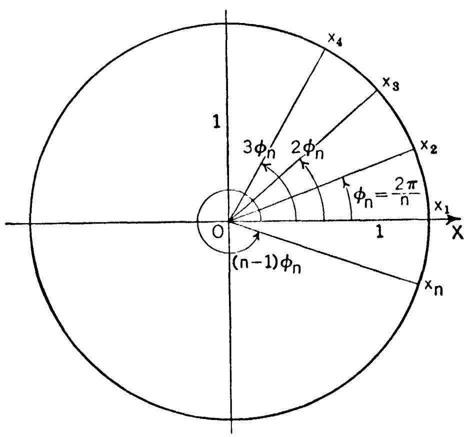


FIG. 79

The representative points of $x_1, x_2, x_3, \cdots x_n$ are obtained as n equally spaced points on a circle of radius 1, the coordinates of the first point being (1, 0) (figure).

To obtain the nth roots of any number a, we need only multiply one of its arithmetic nth roots by the nth roots of unity.

Example.

Find the cube roots of unity.

These are given by $x = \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$; $k = 0, 1, 2.$

$$k = 0; \quad x_1 = \cos 0° + i \sin 0° = 1.$$

$$k = 1; \quad x_2 = \cos 120° + i \sin 120° = -\frac{1}{2} + \frac{i}{2}\sqrt{3}.$$

$$k = 2; \quad x_3 = \cos 240° + i \sin 240° = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$

To find the cube roots of 8, we have $\sqrt[3]{8} = 2\sqrt[3]{1} = 2;\ -1 + i\sqrt{3};\ -1 - i\sqrt{3}$. (We here use $\sqrt[3]{8}$ to denote *any* cube root of 8, not merely the principal root.)

EXERCISES 47

1. Solve the equations $x^3 - 1 = 0$ and $x^3 - 8 = 0$ algebraically and compare with above results.

Solve the following equations by the trigonometric method and give a figure for each case:

2. $x^4 = 1$.
3. $x^4 = 81$.
4. $x^5 = 1$.
5. $x^5 = 32$.
6. $x^6 = 1$.
7. $x^6 = 27$.

109. To express sin $n\theta$ and cos $n\theta$ in terms of powers of sin θ and cos θ, n being a positive integer.

We have $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

Expand the left member by the binomial theorem, reduce all powers of i to ± 1 or $\pm i$, and group the real terms and those involving i. The above equation then becomes

$$\cos n\theta + i\sin n\theta = \left(\cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \cdots\right)$$
$$+ i\left(n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \cdots\right).$$

This equation has the form $P + iQ = P' + iQ'$.

Hence by the corollary in §107 we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \cdots.$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \cdots.$$

Examples.

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

EXERCISES 48

Expand in powers of $\sin\theta$ and $\cos\theta$:

1. $\sin 3\theta$.
2. $\cos 3\theta$.
3. $\cos 4\theta$.
4. $\sin 5\theta$.
5. $\sin 6\theta$.
6. $\cos 7\theta$.

110. Exponential values of sin x and cos x.

We shall assume the following expansions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

These expansions are derived by the methods of Differential Calculus. The letter e stands for an irrational number, $e = 2.7182818+$, which is the base of the *natural* system of logarithms. The last two series are used for calculating $\sin x$ and $\cos x$, by putting for x its value in *radians*. Thus, to calculate $\sin 10°$, put $x = 10° = 0.17453$ radians. (Table V.)

In the first series replace x by ix and *define* the result to be e^{ix}; noting that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \cdots,$$

we obtain

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right). \end{aligned}$$

Hence

$$e^{ix} = \cos x + i \sin x.$$

Replacing x by $-x$;

$$e^{-ix} = \cos x - i \sin x.$$

From these equations we find

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

These formulas are useful in many applications of the trigonometric functions.

EXERCISES

Using the exponential values of $\sin x$ and $\cos x$, show that:

1. $\sin^2 x + \cos^2 x = 1$.

2. $\sin 2x = 2 \sin x \cos x$.

3. $\cos 2x = \cos^2 x - \sin^2 x$.

4. $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x$.

111. The hyperbolic functions.

In the expansions for $\sin x$ and $\cos x$ given at the beginning of §110 replace x by ix and *define* the results to be $\sin ix$ and $\cos ix$ respectively. We obtain

$$\sin ix = i\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right);$$

$$\cos ix = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

These equations we consider as defining the sine and cosine of the imaginary quantity ix.

Multiply the first equation by i and subtract the result from the second. We obtain

$$\cos ix - i \sin ix = e^x.$$

Change x to $-x$; $\cos ix + i \sin ix = e^{-x}$.

Note that, from the definitions of $\cos ix$ and $\sin ix$,

$$\cos(-ix) = \cos ix \text{ and } \sin(-ix) = -\sin ix.$$

Combining the two preceding equations by addition and subtraction, we find

$$\cos ix = \frac{e^x + e^{-x}}{2}; \quad \sin ix = i\frac{e^x - e^{-x}}{2}.$$

We now define

$$\textit{Hyperbolic cosine of } x = \cosh x = \cos ix;$$

$$\textit{Hyperbolic sine of } x = \sinh x = \frac{1}{i} \sin ix.$$

Then

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are related to the hyperbola somewhat as the circular functions to the circle.

Their values can be calculated readily from the values of e^x and e^{-x} given in Table VI.

The remaining hyperbolic functions are defined by the equations

$$\tanh x = \frac{\sinh x}{\cosh x}; \quad \coth x = \frac{1}{\tanh x};$$

$$\operatorname{sech} x = \frac{1}{\cosh x}; \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

EXERCISES

Show that:

1. $\sinh 0 = 0$; $\cosh 0 = 1$.

2. $\sinh \pi i = 0$; $\cosh \pi i = -1$.

3. $\sinh \frac{\pi i}{2} = i$; $\cosh \frac{\pi i}{2} = 0$.

4. $\sinh (-x) = -\sinh x$.

5. $\cosh (-x) = \cosh x$.

6. $\cosh^2 x - \sinh^2 x = 1$.

7. $\operatorname{sech}^2 x = 1 - \tanh^2 x$.

8. $-\operatorname{csch}^2 x = 1 - \coth^2 x$.

Draw the graphs of the equations (see Table VI):

9. $y = e^x$.

10. $y = e^{-x}$.

11. $y = \cosh x$.

12. $y = \sinh x$.

CHAPTER

XI

SPHERICAL TRIGONOMETRY

112. Spherical geometry.

We devote this article to a review of some facts concerning the geometry of the sphere.

(a) A plane section of a sphere is a circle. When the plane passes through the center of the sphere, the section is a *great circle;* otherwise a *small circle.*

(b) Any two great circles intersect in two diametrically opposite points and bisect each other.

(c) The two points on the sphere each equally distant from all the points of a circle on the sphere are called the *poles* of the circle. A great circle is 90° distant from each of its poles.

(d) A *spherical triangle* is a figure bounded by three circular arcs on a sphere. In this chapter we consider only triangles whose sides are arcs of great circles. Any such triangle may therefore be considered as cut from the spherical surface by the faces of a triedral angle whose vertex is at the center. The face angles of this triedral angle measure the sides of the triangle, and its diedral angles the angles of the triangle.

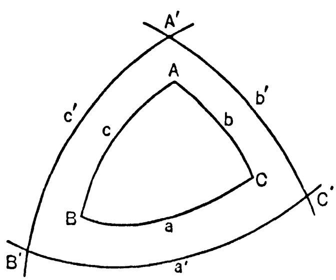
The arcs forming the sides of a spherical triangle will be considered as measured in degrees or in radians. Their lengths in linear units can be obtained if the radius of the sphere is given.

We shall also assume that each side and each angle is less than 180°, in general.

(e) If a triangle be constructed by striking arcs of great circles with the vertices of a given triangle as poles, the new triangle is called the *polar triangle* of the given one.

This method of construction will, in general, yield eight triangles whose vertices are the poles of the given triangle. One of these, and only one, satisfies the following relations.

Let the sides of the given triangle be a, b, c; its angles α, β, γ; let the sides of the polar triangle be a', b', c' and its angles α', β', γ'; we assume that vertex A is the pole of side a'; vertex B of side b'; and vertex C of side c'; then



$$a' = 180° - \alpha;$$
$$\alpha' = 180° - a;$$

and similarly for the other sides and angles. That is, *any part of the polar triangle is the supplement of the part opposite in the given triangle.*

The adjacent figure shows a triangle ABC and its polar triangle $A'B'C'$; A is the pole of arc $B'C'$, B of arc $A'C'$, C of arc $A'B'$.

(f) The sum of the angles of a spherical triangle is greater than 180° and less than 540°. The amount by which the angle sum exceeds 180° is called the *spherical excess* of the triangle. Two formulas for calculation of the spherical excess are given in §126.

The area of a spherical triangle is to the area of the sphere as its spherical excess, in degrees, is to 720°. That is, if E be the spherical excess in degrees and K the area of the triangle, and R the radius of the sphere, then

$$\frac{K}{4\pi R^2} = \frac{E}{720}; \quad \text{or} \quad K = 4\pi R^2 \frac{E}{720}.$$

(g) The sum of the sides of a spherical triangle is less than 360°.

113. The terrestrial sphere.

To illustrate some of the definitions just given we shall relate them to the surface of the earth considered as a sphere with radius $R = 3960$ miles.

The earth's axis of rotation meets the surface at two points, P and P', the *north geographical pole* and the *south geographical pole*.

A plane through the center of the earth and perpendicular to axis PP' cuts the surface in a great circle called the *equator*.

A plane perpendicular to axis PP' at any point between P and P' other than the midpoint cuts the surface in a small circle called a *parallel of latitude*. The tropics (Cancer and Capricorn) and the two arctic circles are such parallels.

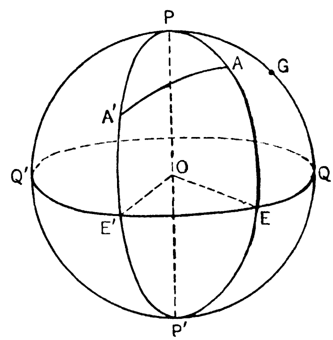


FIG. 81

Any plane which contains the axis of rotation PP' meets the surface in a great circle called a *meridian*.

Any meridian cuts the equator in two diametrically opposite points. For the "*prime meridian*" (meridian of Greenwich, PGQ) these are the points on the equator of 0° longitude and 180° longitude, Q and Q', respectively.

If A is a station on the earth's surface on meridian PAE, arc EA = *latitude* of A and angle QPA = *longitude* of A.

Latitude is counted positive when point A is north of the equator and counted negative when point A is south of the equator.

Arc PA is the *north polar distance* of A and is counted from 0° to 180°. It is the complement of the latitude, and is greater than 90° when the latitude is negative.

If A' is a second station, $E'A'$ = latitude of A', angle QPA' = longitude of A', the angle $A'PA$ = the difference of longitude, DLO, of A and A'.

If a plane be passed through the earth's center O and points A and A', the plane will cut the earth's surface in the great circle AA'.

Any other plane containing points AA' will cut the earth's surface in a " small " circle.

The shortest distance between A and A' is the distance measured along the great circle joining the points.

The spherical triangle APA', whose vertices are two stations on the earth's surface and the north pole, is much used in the applications of spherical trigonometry. If the latitudes and longitudes of A and A' are given, we know also their polar distances; that is, the sides AP and $A'P$ of the triangle. The difference of longitudes is the angle APA' included between these sides.

The determination of the remaining parts of triangle APA', when two sides and the included angle are given, constitutes a basic problem of spherical trigonometry. If an airplane is to fly from A to A' by the shortest route, it would have to start from point A at an angle PAA' with the true north.

114. Spherical right triangles.

Let O be the center of a sphere and ABC a triangle on its surface, with the angle at C equal to 90°.

It should be noted that a spherical triangle may have two, or even three, right angles. When there is more than one right angle the side opposite each right angle is a quadrant.

We shall use small letters a, b, c to indicate the sides opposite the vertices A, B, C, respectively.

The angles of the triangle, at vertices A, B, C, we shall indicate by the Greek letters α, β, γ, respectively. Therefore $\gamma = 90°$.

Figure 82 indicates such a triangle, side AC being an arc of a great circle which we might think of as the equator and side CB then being an arc of a meridian. The right angle is at C and AB is the hypotenuse.

Such a triangle is again represented in Fig. 83. In this figure

pass a plane perpendicular to OA at A' and let this plane meet OB in B' and OC in C'. The plane angle $B'A'C'$ measures angle $BAC = \alpha$ of the spherical triangle. §112(d).

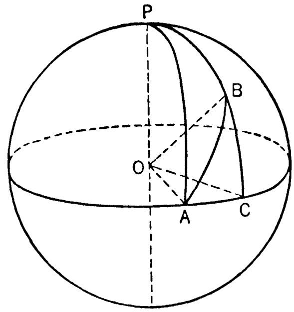


FIG. 82

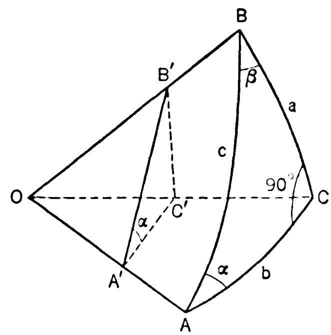


FIG. 83

The following triangles are plane right triangles:

$\triangle\ OA'B'$; rt. angle at A'; $\triangle\ OA'C'$; rt. angle at A';
$\triangle\ A'C'B'$; rt. angle at C'; $\triangle\ OC'B'$; rt. angle at C'.

Then from plane trigonometry,

(a) $$\sin\,\alpha = \sin\,B'A'C' = \frac{C'B'}{A'B'} = \frac{\dfrac{C'B'}{OB'}}{\dfrac{A'B'}{OB'}} = \frac{\sin a}{\sin c}.$$

(b) $$\cos\,\alpha = \cos\,B'A'C' = \frac{A'C'}{A'B'} = \frac{\dfrac{A'C'}{OA'}}{\dfrac{A'B'}{OA'}} = \frac{\tan b}{\tan c}.$$

(c) $$\tan\,\alpha = \tan\,B'A'C' = \frac{B'C'}{A'C'} = \frac{\dfrac{B'C'}{OC'}}{\dfrac{A'C'}{OC'}} = \frac{\tan a}{\sin b}.$$

Dividing (a) by (b) and comparing with (c) we have

(d) $$\cos c = \cos a \cos b.$$

Interchanging a with b and α with β in (a), (b), (c) gives three similar formulas, making seven relations.

These may be combined to give three additional formulas, making ten in all. They are stated below, in forms cleared of fractions.

(1)	$\sin a = \sin c \sin \alpha,$	(6)	$\sin b = \sin c \sin \beta,$
(2)	$\tan b = \tan c \cos \alpha,$	(7)	$\tan a = \tan c \cos \beta,$
(3)	$\tan a = \sin b \tan \alpha,$	(8)	$\tan b = \sin a \tan \beta,$
(4)	$\cos c = \cos a \cos b,$	(9)	$\cos c = \cot \alpha \cot \beta,$
(5)	$\cos \alpha = \cos a \sin \beta,$	(10)	$\cos \beta = \cos b \sin \alpha.$

Here (1), (2), (3), (4) are (a), (b), (c), (d) cleared of fractions; from (1), (2), (3) we obtain (6), (7), (8) by interchange of letters.

To obtain formula (5) solve (3) for $\cos \alpha$, obtaining

$$\begin{aligned} \cos \alpha &= \sin \alpha \cot a \cdot \sin b \\ &= \sin \alpha \cot a \cdot \sin c \sin \beta && \text{from (6)} \\ &= \sin a \cot a \sin \beta && \text{from (1)} \\ &= \cos a \sin \beta. \end{aligned}$$

Formula (10) results from (5) by interchange of letters.

To obtain (9), solve (3) for $\cos a$, solve (8) for $\cos b$, and substitute these in (4).

115. Napier's rules of circular parts.

Let co-x denote the complement of any part x of the triangle. Take the complements of c, α, β, and arrange the five parts, a, b, co-α, co-c, co-β, called *circular parts*, in the order in which they occur in the triangle, as in the adjacent figures. Then if

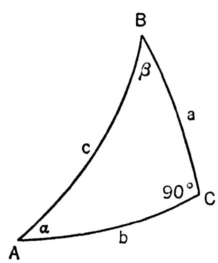


FIG. 84

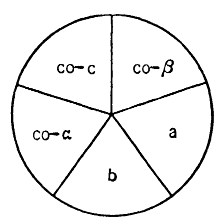


FIG. 85

any one of the five be taken as the middle part, of the other four parts two will be *adjacent* and the other two *opposite* to this part. Thus, if co-c be taken as the middle part, co-β and co-α are adjacent, a and b opposite.

If c exceeds 90° co-c will be negative; similarly for α and β.

Napier's Rules:

$$\textit{Sine of Middle Part} = \begin{cases} \textit{Product of tangents of adjacent parts,} \\ \textit{or} \\ \textit{Product of cosines of opposite parts.} \end{cases}$$

Example.

With co-c as middle part Napier's rules give

$$\sin(\text{co-}c) = \tan(\text{co-}\alpha)\tan(\text{co-}\beta) \quad \text{or} \quad \cos c = \cot\alpha \cot\beta;$$
$$\sin(\text{co-}c) = \cos a \cos b \quad \text{or} \quad \cos c = \cos a \cos b.$$

These are formulas (4) and (9).

Exercise. Taking each part in turn as the middle part write out a complete list of formulas relating to the spherical right triangle.

116. Solution of right spherical triangles.

When two parts of a right triangle are given, in addition to the right angle, we can always apply Napier's rules to write down three equations each of which contains the two given parts and one of the unknown parts. These equations then determine the three unknown parts.

Ambiguous Case. When an unknown part is determined by the value of its sine, two supplementary values are obtained, and there may be two solutions. This happens when the given parts are an angle and its opposite side, α and a or β and b.

In this case the two triangles determined by the two solutions together form a lune, as AA' in Fig. 86, where the given parts are assumed to be angle α with vertex at A and its opposite side a.

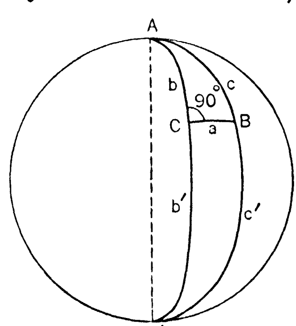


FIG. 86

When an unknown part is determined by its cosine or tangent there is no ambiguity. If the function is positive, the part lies in the first quadrant; if negative, in the second quadrant.

In the ambiguous case care must be taken to select the three unknown parts properly from the three pairs of answers. As a guide to this selection the following rules will be useful.

1. The sum of two sides must be greater than the third side.

2. If two sides are unequal, the opposite angles are unequal, and the greater angle lies opposite the greater side.

3. Half the sum of two sides is in the same quadrant as half the sum of their opposite angles.

4. Sides a and b are in the same quadrant if side c is in quadrant I; they are in different quadrants if side c is in quadrant II.

5. A side and its opposite angle are in the same quadrant.

Rules 4 and 5 are easily obtained by inspection of the ten formulas. Rule 4 follows from formulas (2) and (7) and rule 5 from (3) and (8). The first three rules apply also to oblique spherical triangles.

117. *Examples.*

We shall consider several examples, of which the second illustrates the ambiguous case.

In writing logarithms having characteristic 9–10 the –10 is omitted to save space.

For a check use Napier's rules to write an equation containing the three unknown parts.

Example 1.

Given $a = 35° 42'$; $\beta = 60° 25'$. Find b, c, α.

The diagram of circular parts is shown in the figure. Taking (1), (2), (3) in turn as middle part we have

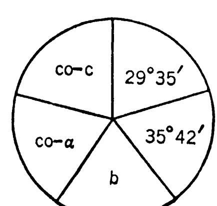


FIG. 87

(1) $\sin 35° 42' = \tan 29° 35' \tan b;$
(2) $\sin 29° 35' = \tan 35° 42' \tan (\text{co-}c);$
(3) $\sin (\text{co-}\alpha) = \cos 29° 35' \cos 35° 42'.$

Hence

$$\tan b = \frac{\sin 35^\circ 42'}{\tan 29^\circ 35'}; \quad \cot c = \frac{\sin 29^\circ 35'}{\tan 35^\circ 42'};$$

$$\cos \alpha = \cos 29^\circ 35' \cos 35^\circ 42'.$$

Check. The computed parts must satisfy the relation

$$\sin(\text{co-}\alpha) = \tan b \tan(\text{co-}c), \quad \text{or} \quad \cos \alpha = \tan b \cot c.$$

Computations.

log		log		log	
$\sin 35^\circ 42'$ =	9.7660	$\sin 29^\circ 35'$ =	9.6934	$\cos 29^\circ 35'$ =	9.9394
$\tan 29^\circ 35'$ =	9.7541	$\tan 35^\circ 42'$ =	9.8564	$\cos 35^\circ 42'$ =	9.9096
$\tan b$ =	0.0119	$\cot c$ =	9.8370	$\cos \alpha$ =	9.8490
b =	$45^\circ 17'$	c =	$55^\circ 30'$	α =	$45^\circ 4'$

Check. $\log \cos \alpha = \log \tan b + \log \cot c.$

$$9.8490 = 0.0119 + 9.8370.$$

Example 2.

Given $\alpha = 48^\circ 25'$, $a = 32^\circ 13'$. Find b, c, β.

Using (1), (2), (3) in turn as middle part, Napier's rules give

(1) $\sin b = \tan 41^\circ 35' \tan 32^\circ 13'$;
(2) $\sin 41^\circ 35' = \cos(\text{co-}\beta) \cos 32^\circ 13'$;
(3) $\sin 32^\circ 13' = \cos(\text{co-}c) \cos 41^\circ 35'$.

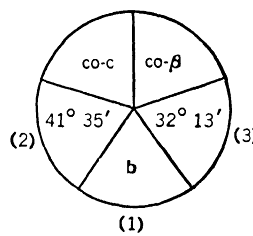


FIG. 88

Solving for the unknown parts:

$$\sin b = \tan 41^\circ 35' \tan 32^\circ 13';$$

$$\sin \beta = \frac{\sin 41^\circ 35'}{\cos 32^\circ 13'}.$$

$$\sin c = \frac{\sin 32^\circ 13'}{\cos 41^\circ 35'}.$$

Check. $\sin b = \cos(\text{co-}c) \cos(\text{co-}\beta) = \sin c \sin \beta.$

Computations.

log		log		log	
$\tan 41^\circ 35'$ =	9.9481	$\sin 41^\circ 35'$ =	9.8220	$\sin 32^\circ 13'$ =	9.7268
$\tan 32^\circ 13'$ =	9.7994	$\cos 32^\circ 13'$ =	9.9274	$\cos 41^\circ 35'$ =	9.8739
$\log \sin b$ =	9.7475	$\log \sin \beta$ =	9.8946	$\log \sin c$ =	9.8529
b =	$33^\circ 59'$	β =	$51^\circ 41'$	c =	$45^\circ 27'$
b' =	$146^\circ 1'$	β' =	$128^\circ 19'$	c' =	$134^\circ 33'$

Check. $\log \sin b = \log \sin c + \log \sin \beta.$

$$9.7475 = 9.8529 + 9.8946.$$

If the logarithm of the sine of one of the unknown parts is 0, that part is 90°, and there is only one solution. If the logarithm is positive there is no solution.

Example 3.

Given $a = 50°$, $c = 120°$. Find side b.

Here co-$c = -30°$, a negative angle. To obtain side b Napier's Rules give, with co-c as middle part,

$$\sin(-30°) = \cos b \cos 50°, \quad \text{or,} \quad \cos b = \sin(-30°) \sec 50°.$$

Since $\sin(-30°)$ is a negative number $\cos b$ is negative and b is in quadrant II. We obtain

$$\cos b = -\tfrac{1}{2} \times 1.5557 = -0.7778. \quad b = 180° - 38°\,56' = 141°\,4'.$$

Example 4.

Solution of an oblique spherical triangle.

In triangle ABC let there be given two sides and their included angle, namely

$$b = 63°\,22',\ c = 59°\,17',\ \alpha = 81°\,39'.$$

The unknown parts, side a and angles β and γ are to be calculated.

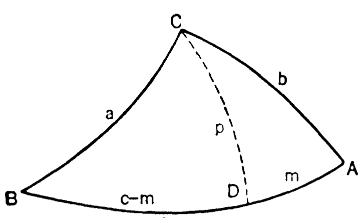


FIG. 89a

Divide the oblique triangle into two right triangles by the perpendicular CD drawn from vertex C on side AB, as in Fig. 89a. Let $p =$ arc CD, $m =$ arc AD, $c - m =$ arc DB. In right triangle CDA side b and angle α are known so that we can calculate p, m, and angle DCA. Then in right triangle CDB we know p and $c - m$, and can calculate side a, angle β and angle DCB. Finally the sum of angle DCA and angle DCB equals angle γ.

The student should note the close analogy between the method used here and that used in the corresponding problem for the plane oblique triangles. See Example 1 at the end of §43.

EXERCISE

(a) Show that in $\triangle\, CDA$ Napier's Rules give,

with co-b as middle part; $\cos b = \cot \alpha \cot DCA$,
or, $\cot DCA = \cos b \tan \alpha$:
with co-α as middle part; $\cos \alpha = \cot b \tan m$,
or, $\tan m = \tan b \cos \alpha$:
with p as middle part; $\sin p = \sin b \sin \alpha$.

(b) Show that in $\triangle\, CDB$ Napier's Rules give,

with p as middle part; $\sin p = \tan (c - m) \cot DCB$, or, $\cot DCB = \cot (c - m) \sin p$:
with $c - m$ as middle part; $\sin (c - m) = \cot \beta \tan p$, or, $\cot \beta = \sin (c - m) \cot p$:
with co-a as middle part; $\cos a = \cos (c - m) \cos p$.

(c) Use the numerical values given above to calculate angle DCA, m and p by the formulas under (a), then angle DCB, β, and a by the formulas under (b), and finally angle γ. Use 4-place tables.

Ans. $a = 70° 7'$, $\beta = 70° 9'$, $\gamma = 64° 49'$.

118. Quadrantal triangles.

A quadrantal triangle is one having a side equal to a quadrant or 90°. Its polar triangle will be a right triangle, which may be solved by Napier's Rules. The parts of the given quadrantal triangle then become known by (e) of §112.

119. EXERCISES 49

Solve the following triangles, γ being the right angle:

1. $a = 137° 59'$, $b = 58° 40'$.
2. $a = 137° 50'$, $c = 64° 40'$.
3. $\alpha = 5° 47'$, $\beta = 85° 52'$.
4. $a = 41°$, $\beta = 37°$.
5. $a = 134° 30'$, $c = 122° 8'$.
6. $c = 137° 20'$, $\alpha = 149° 40'$.
7. $c = 73° 35'$, $\beta = 101° 13'$.
8. $a = 74° 7'$, $\alpha = 75° 6'$.
9. $b = 122° 38'$, $\beta = 134° 30'$.
10. $b = 60° 11.4'$, $c = 83° 30.8'$.
11. $c = 129° 14.7'$, $\alpha = 43° 15.7'$.
12. $\alpha = 58° 3.5'$, $\beta = 36° 35.6'$.

Solve the following quadrantal triangles, side c being 90°:

13. $a = 116° 45'$, $b = 44° 26'$.
14. $b = 36° 10'$, $\gamma = 65° 28'$.
15. $a = 18° 8'$, $\beta = 48° 52'$.
16. $b = 35° 6'$, $\beta = 33° 28'$.
17. $a = 108° 23'$, $\gamma = 88° 18'$.
18. $a = 80° 10'$, $\alpha = 68° 0'$.
19. $\beta = 24° 12.6'$, $\gamma = 152° 50.6'$.
20. $a = 58° 52.1'$, $\gamma = 146° 59.4'$.
21. $b = 127° 24.3'$, $\beta = 135° 56.2'$.

120. Oblique spherical triangles. Two fundamental formulas.

I. *Law of sines.*

Let triangle ABC be a spherical oblique triangle. To obtain relations between the parts of such a triangle we draw an arc through a vertex perpendicular to the opposite side and use the resulting right triangles.

The foot of the perpendicular from C on AB, point D, may fall on side c (Fig. 89a) or on side c produced (Fig. 89b).

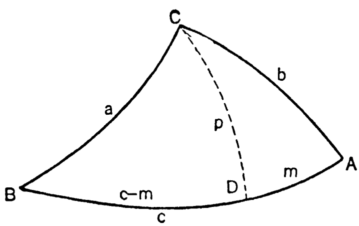


FIG. 89a

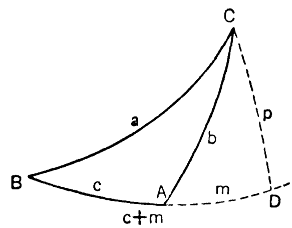


FIG. 89b

By use of Napier's rules:

$\triangle\ ADC$, $\sin p = \sin b \sin \alpha$; $\sin p = \sin b \sin (\pi - \alpha)$;
$\triangle\ BDC$, $\sin p = \sin a \sin \beta$; $\sin p = \sin a \sin \beta$.

But $\sin (\pi - \alpha) = \sin \alpha$. Therefore the equations from Fig. 89b reduce to those for Fig. 89a.

Equating the values of $\sin p$, we have

$$\sin b \sin \alpha = \sin a \sin \beta.$$

This may be written

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta}.$$

By drawing the perpendicular through vertex B a third ratio is introduced and we have

$$(1) \qquad \frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

These relations are known as the law of sines. In verbal form, *the sines of the sides are proportional to the sines of their opposite angles.*

II. ***Law of cosines for sides.***

In Fig. 89a or 89b, let $AD = m$. Then $BD = c - m$, Fig. 89a, and $BD = c + m$, Fig. 89b. We first consider Fig. 89a.

In right $\triangle$ BDC: $\cos a = \cos (c - m) \cos p$

$$= \cos c \cos m \cos p + \sin c \sin m \cos p.$$

We substitute here the values of $\cos m \cos p$ and $\sin m \cos p$ from $\triangle$ ADC.

In $\triangle$ ADC: $\qquad \cos b = \cos m \cos p.$

Also, $\quad \sin m = \sin b \sin ACD \quad$ and $\quad \cos \alpha = \cos p \sin ACD.$

$$\therefore \sin m \cos p = \sin b \cos \alpha.$$

Substituting these in the expression for $\cos a$ we have

(2) $$\boldsymbol{\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.}$$

In Fig. 89b, $BD = c + m$. Also angle $DAC = \pi - \alpha$.

In right $\triangle$ BDC: $\cos a = \cos (c + m) \cos p$

$$= \cos c \cos m \cos p - \sin c \sin m \cos p.$$

In right $\triangle$ ADC: $\cos m \cos p = \cos b$;

$$\begin{aligned} \sin m \cos p &= \sin ACD \sin b \cdot \frac{\cos (\pi - \alpha)}{\sin ACD} \\ &= \sin ACD \sin b \cdot \frac{-\cos \alpha}{\sin ACD} \\ &= -\sin b \cos \alpha. \end{aligned}$$

Substituting in the expression for $\cos a$ we obtain formula (2) exactly as before.

By drawing perpendiculars on the other two sides we would obtain corresponding formulas for those sides. Instead of writing these formulas out separately we include all three in a verbal statement of the *law of cosines for sides*.

The cosine of any side equals the product of the cosines of the other two sides plus the product of their sines by the cosine of their included angle.

From the fundamental formulas (1) and (2) we shall derive a group of other formulas adapted to the solution of spherical triangles.

121. Principle of duality.

By means of (e) of §112 any formula relating to the spherical triangle can be made to yield a second formula. Thus, let $\triangle A'B'C'$ be polar to $\triangle ABC$. Then from (1) and (2), applied to $\triangle A'B'C'$, we have

$$\frac{\sin a'}{\sin b'} = \frac{\sin \alpha'}{\sin \beta'}; \quad \cos a' = \cos b' \cos c' + \sin b' \sin c' \cos \alpha'.$$

But

$$\begin{aligned} a' &= 180° - \alpha, & \alpha' &= 180° - a, \\ b' &= 180° - \beta, & \beta' &= 180° - b, \\ c' &= 180° - \gamma, & \gamma' &= 180° - c. \end{aligned}$$

Substituting and reducing, we have

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin a}{\sin b},$$

$$(3) \qquad \boldsymbol{\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a.}$$

The first of these is simply the law of sines; the second is a new formula. It is called the *law of cosines for angles.*

122. Formulas for the half-angles.

From the half-angle formulas of group C, §73, we have

$$\sin \tfrac{1}{2}\alpha = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$$

Since $\frac{1}{2}\alpha$ is less than 90°, α being less than 180°, we take the + sign.

We work out a value for $1 - \cos \alpha$ to substitute under the radical.

Solving (2) for $\cos \alpha$, we have

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

$$\begin{aligned} 1 - \cos \alpha &= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c} \\ &= \frac{\cos (b - c) - \cos a}{\sin b \sin c} && \text{§67} \\ &= \frac{-2 \sin \frac{1}{2}(b - c + a) \sin \frac{1}{2}(b - c - a)}{\sin b \sin c}. && \text{§75} \end{aligned}$$

$$\frac{1 - \cos\alpha}{2} = \frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b \sin c}. \qquad \S 23$$

(4) Let $\quad 2s = a + b + c.$

Then $\quad \frac{1}{2}(a+b-c) = s - c; \quad \frac{1}{2}(a-b+c) = s - b. \qquad \S 82$

Therefore

$$(5) \qquad \sin\tfrac{1}{2}\alpha = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}.$$

Similarly, starting with $\cos\frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1 + \cos\alpha)}$, we get

$$(6) \qquad \cos\tfrac{1}{2}\alpha = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

By dividing,

$$(7) \qquad \tan\tfrac{1}{2}\alpha = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

Given the three sides, one of these formulas, preferably the last, will determine the angles. When all three angles are desired, let

$$(8) \qquad \tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

then

$$(9) \qquad \tan\tfrac{1}{2}\alpha = \frac{\tan r}{\sin(s-a)},$$

$$(10) \qquad \tan\tfrac{1}{2}\beta = \frac{\tan r}{\sin(s-b)},$$

$$(11) \qquad \tan\tfrac{1}{2}\gamma = \frac{\tan r}{\sin(s-c)}.$$

123. Formulas for the half sides.

Proceeding as above with (3) of §121, or by applying the principle of duality to formulas (5) to (11), we have, on putting

$$(12) \qquad 2S = \alpha + \beta + \gamma$$

and

$$(13) \qquad \tan R = \sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}},$$

$$(14) \qquad \sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-\alpha)}{\sin\beta\sin\gamma}},$$

$$(15) \qquad \cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-\beta)\cos(S-\gamma)}{\sin\beta\sin\gamma}},$$

$$(16)\qquad \tan \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S-\alpha)}{\cos (S-\beta) \cos (S-\gamma)}},$$

$$(17)\qquad \tan \tfrac{1}{2}a = \tan R \cos (S-\alpha),$$

$$(18)\qquad \tan \tfrac{1}{2}b = \tan R \cos (S-\beta),$$

$$(19)\qquad \tan \tfrac{1}{2}c = \tan R \cos (S-\gamma).$$

124. Napier's analogies.

Dividing $\tan \frac{1}{2}\alpha$ by $\tan \frac{1}{2}\beta$ we obtain

$$\frac{\tan \frac{1}{2}\alpha}{\tan \frac{1}{2}\beta} = \frac{\sin (s-b)}{\sin (s-a)}.$$

By composition and division, or by following the steps in the first part of §81, we obtain

$$\text{(a)}\qquad \frac{\tan \frac{1}{2}\alpha + \tan \frac{1}{2}\beta}{\tan \frac{1}{2}\alpha - \tan \frac{1}{2}\beta} = \frac{\sin (s-b) + \sin (s-a)}{\sin (s-b) - \sin (s-a)}.$$

To reduce the fraction on the left we write, for convenience,

$$x = \tfrac{1}{2}\alpha, \quad y = \tfrac{1}{2}\beta.$$

Then

$$\frac{\tan x + \tan y}{\tan x - \tan y} = \frac{\tan x + \tan y}{\tan x - \tan y} \cdot \frac{\cos x \cos y}{\cos x \cos y}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y} = \frac{\sin (x+y)}{\sin (x-y)}.$$

$$\frac{\tan \frac{1}{2}\alpha + \tan \frac{1}{2}\beta}{\tan \frac{1}{2}\alpha - \tan \frac{1}{2}\beta} = \frac{\sin \frac{1}{2}(\alpha+\beta)}{\sin \frac{1}{2}(\alpha-\beta)}.$$

To reduce the fraction on the right side of equation (a) we write $u = s - b$ and $v = s - a$. Then, §67,

$$\frac{\sin u + \sin v}{\sin u - \sin v} = \frac{2 \sin \frac{1}{2}(u+v) \cos \frac{1}{2}(u-v)}{2 \cos \frac{1}{2}(u+v) \sin \frac{1}{2}(u-v)}$$

$$= \frac{\tan \frac{1}{2}(u+v)}{\tan \frac{1}{2}(u-v)}.$$

But

$$u + v = s - b + s - a = 2s - a - b = c;$$

$$u - v = s - b - s + a = a - \mathrm{b}.$$

$$\frac{\sin (s-b) + \sin (s-a)}{\sin (s-b) - \sin (s-a)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a-b)}.$$

Then equation (a) reduces to

$$\text{(20)} \qquad \frac{\sin \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a - b)}.$$

Similar formulas may be obtained involving the pairs of angles α, γ and β, γ. All may be expressed by the same verbal statement.

In applications to the solution of triangles, (20) is written in the form

$$\text{(20}'\text{)} \qquad \tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)} \tan \tfrac{1}{2}c.$$

Multiplying $\tan \frac{1}{2}\alpha$ by $\tan \frac{1}{2}\beta$ and reducing,

$$\frac{\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta}{1} = \frac{\sin (s - c)}{\sin s}.$$

By composition and division, and reduction as above,

$$\text{(21)} \qquad \frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a + b)},$$

or

$$\text{(21}'\text{)} \qquad \tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)} \tan \tfrac{1}{2}c.$$

These formulas determine the other two sides when two angles and their included side are given.

Proceeding as above with $\tan \frac{1}{2}a$ and $\tan \frac{1}{2}b$, or by the principle of duality applied to formulas (20) and (21), we obtain

$$\text{(22)} \qquad \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} = \frac{\cot \frac{1}{2}\gamma}{\tan \frac{1}{2}(\alpha - \beta)},$$

or

$$\text{(22}'\text{)} \qquad \tan \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}\gamma;$$

and

$$\text{(23)} \qquad \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)} = \frac{\cot \frac{1}{2}\gamma}{\tan \frac{1}{2}(\alpha + \beta)},$$

or

$$\text{(23}'\text{)} \qquad \tan \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}\gamma.$$

These formulas determine the other two angles when two sides and their included angle are given.

125. Delambre's or Gauss's analogies.

These are formulas for the sine and cosine of the half-sum and the half-difference of two angles.

$$\sin \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}c} \cos \tfrac{1}{2}\gamma; \tag{24}$$

$$\sin \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}c} \cos \tfrac{1}{2}\gamma; \tag{25}$$

$$\cos \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}c} \sin \tfrac{1}{2}\gamma; \tag{26}$$

$$\cos \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}c} \sin \tfrac{1}{2}\gamma. \tag{27}$$

We shall show how (27) is derived.

$$\cos \tfrac{1}{2}(\alpha - \beta) = \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta + \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta.$$

From the half-angle formulas we obtain

$$\frac{\cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta}{\sin \frac{1}{2}\gamma} = \frac{\sin s}{\sin c}; \quad \frac{\sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta}{\sin \frac{1}{2}\gamma} = \frac{\sin (s - c)}{\sin c}.$$

Adding these we have

$$\begin{aligned} \frac{\cos \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}\gamma} &= \frac{\sin s + \sin (s - c)}{\sin c} \\ &= \frac{2 \sin \frac{1}{2}(2s - c) \cos \frac{1}{2}c}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} \\ &= \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}c}. \end{aligned}$$

Multiplying both sides by $\sin \frac{1}{2}\gamma$ gives (27).

126. Area of a spherical triangle.

This may be calculated by (f) of (82), namely,

$$K = \frac{E \text{ (degrees)}}{720} \times 4\pi R^2, \quad \text{or,} \quad K = E \text{ (radians)} \times R^2.$$

To obtain E, we may first calculate the angles. E may also be obtained by one of the following formulas which we add without proofs.

$$\tan \tfrac{1}{2}E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin \gamma}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos \gamma};$$

$$\tan \tfrac{1}{4}E = \sqrt{\tan \frac{s}{2} \tan \frac{s - a}{2} \tan \frac{s - b}{2} \tan \frac{s - c}{2}}.$$

127. Solution of spherical oblique triangles.

Six cases arise, according to the nature of the three given parts.

I. Given two sides and an opposite angle.

Denote the given parts by a, b, α. Calculate β by (1), then γ by (22) or (23), and c by (20) or (21).

Check. $\sin b : \sin c = \sin \beta : \sin \gamma,$

which involves the *computed parts.*

Ambiguous Case. Formula (1) will give two (supplementary) values for β. Two solutions are obtained when both values of β lead to values of γ. Otherwise one or both values of β must be rejected.

Rule. Retain values of β which make $\alpha - \beta$ and $a - b$ of like sign.

Otherwise (20) and (22) take the impossible form $+ = -$.

II. Given two angles and an opposite side.

Denote the given parts by α, β, a. Calculate b by (1), then proceed as in I.

Ambiguous Case. Formula (1) gives two values of b. *Retain the value or values which make $\alpha - \beta$ and $a - b$ of like sign.*

III. Given the three sides.

Calculate the angles by (9), (10), (11).

Check. $\sin \alpha : \sin a = \sin \beta : \sin b = \sin \gamma : \sin c.$

IV. Given the three angles.

Calculate the sides by (17), (18), (19).

Check. As in III.

V. Given two sides and their included angle.

Denote the given parts by a, b, γ. Calculate $\frac{1}{2}(\alpha + \beta)$ by (23′), $\frac{1}{2}(\alpha - \beta)$ by (22′); then α and β by addition and subtraction; obtain c in two ways by the law of sines. This furnishes a check; or check by (20) or (21).

VI. Given two angles and their included side.

Denote the given parts by α, β, c. Calculate $\frac{1}{2}(a+b)$ from (21′), $\frac{1}{2}(a-b)$ from (20′); hence get a and b; obtain γ in two ways by the law of sines. This gives a check; or check by (22) or (23).

The quadrant of a side or angle, when in doubt, may often be decided readily by the use of Rules 1, 2, or 3 of §116. These three rules apply to oblique triangles as well as to right triangles.

128. Alternative method under Case V.

When two sides and their included angle are given, each of the unknown parts can be calculated independently by compact formulas well adapted to logarithmic computation. These formulas will now be derived. Applications will be given in the next chapter. See also the note in §134.

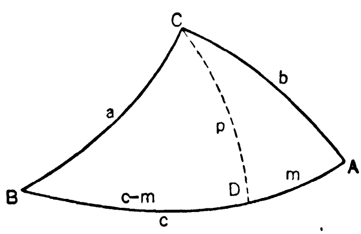


FIG. 89a

Case V.

Given b, c, α. To determine a, β, γ.

We return to Fig. 89a, which is reproduced here for convenience of reference. The case of Fig. 89b will be discussed later.

Apply Napier's Rules to triangle CDA:

1) $\cos b = \cos m \cos p$, or $\cos p = \cos b \sec m$.
2) $\sin m = \tan p \cot \alpha$, or $\cot p = \cot \alpha \csc m$.
3) $\cos \alpha = \tan m \cot b$, or $\tan m = \tan b \cos \alpha$.

Apply Napier's Rules to triangle CDB:

4) $\cos a = \cos p \cos (c - m)$.
5) $\sin (c - m) = \tan p \cot \beta$, or $\cot \beta = \cot p \sin (c - m)$.

Substitute $\cos p$ and $\cot p$ from 1) and 2) in 4) and 5):

6) $\cos a = \cos b \sec m \cos (c - m)$.
7) $\cot \beta = \cot \alpha \csc m \sin (c - m)$.

Equation 3) gives m, 6) gives a, 7) gives β.

To obtain γ = angle BCA, we may suppose a perpendicular BD' to be drawn from B to side AC, and let $AD' = n$. Then we obtain, in place of 3), 6), 7), the following equations:

3′) $\tan n = \tan c \cos \alpha$.
6′) $\cos a = \cos c \sec n \cos (b - n)$.
7′) $\cot \gamma = \cot \alpha \csc n \sin (b - n)$.

As to the case of Fig. 89b, if we regard arc m as a positive length, then arc $DB = c + m$ and this quantity would appear in

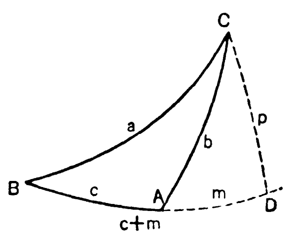


FIG. 89b

6) and 7). But if we regard m as a signed quantity we see from 3) that m will change sign when angle α becomes obtuse and so we must write arc $DB = c - m$, not $c + m$. Hence we obtain the same formulas from either figure.

For convenience of reference we group the formulas of this section.

Alternative formulas for Case V. Given b, c, α.

(28) $\tan m = \tan b \cos \alpha$; $\tan n = \tan c \cos \alpha$.
(29) $\cos a = \cos b \sec m \cos (c - m) = \cos c \sec n \cos (b - n)$.
(30) $\cot \beta = \cot \alpha \csc m \sin (c - m)$.
(31) $\cot \gamma = \cot \alpha \csc n \sin (b - n)$.

129. Haversine Formulas.

The haversine function, defined in §25, may be conveniently employed when three sides of a triangle are given and only one of the angles is required, or when two sides and their included angle are given and the third side is required. Extensive tables of this function have been calculated. A brief table is included in Appendix B.

(a) *Given the three sides, to find one of the angles.*

The square of the half-angle formula (20) gives

$$\sin^2 \tfrac{1}{2}\alpha = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}.$$

But $\qquad \sin^2 \tfrac{1}{2}\alpha = \dfrac{1 - \cos \alpha}{2} = \operatorname{hav} \alpha.$

Therefore

$$\textbf{hav}\ \alpha = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}. \tag{32}$$

(b) The same problem may be solved by starting with the law of cosines and introducing the haversine function.

From the law of cosines:

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

$$\begin{aligned} 1 - \cos \alpha &= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c} \\ &= \frac{\cos (b - c) - \cos a}{\sin b \sin c}. \end{aligned}$$

But

$$\begin{aligned} 1 - \cos \alpha &= 2 \operatorname{hav} \alpha; \\ \cos (b - c) &= 1 - 2 \operatorname{hav} (b - c); \\ \cos a &= 1 - 2 \operatorname{hav} a. \end{aligned}$$

Therefore

$$\textbf{hav}\ \alpha = \frac{\operatorname{hav} a - \operatorname{hav} (b - c)}{\sin b \sin c}. \tag{33}$$

(c) A frequently occurring problem in the applications is the calculation of the third side of a triangle from two sides and their included angle. It is the problem involved in finding the distance between two stations whose latitudes and longitudes are known.

The haversine formula for this problem is obtained directly from (33) by solving for hav a.

(34) $$\textit{hav } a = \textit{hav } (b - c) + \sin b \sin c \textit{ hav } \alpha.$$

Examples of the use of these formulas will be found in the following chapter.

130. Suggested forms for computations.

Case I. Given two sides and an opposite angle.

Example.

Given $a = 100°\ 37'$, $b = 62°\ 25'$, $\alpha = 120°\ 48'$.

Formulas.
$$\sin \beta = \frac{\sin b}{\sin a} \sin \alpha,$$

$$\cot \tfrac{1}{2}\gamma = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(\alpha - \beta),$$

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(\alpha+\beta)}{\sin \frac{1}{2}(\alpha-\beta)} \tan \tfrac{1}{2}(a - b).$$

Check.
$$\frac{\sin b}{\sin c} = \frac{\sin \beta}{\sin \gamma}.$$

Computations.

$\log \sin b = 9.9476$	$a = 100°\ 37'$	$\alpha = 120°\ 48'$
$\log \sin \alpha = 9.9340$	$b = 62°\ 25'$	$\beta = 50°\ 46'$
$\text{colog} \sin a = 0.0075$	$a + b = 162°\ 62'$	$\alpha + \beta = 170°\ 94'$
$\log \sin \beta = 9.8891$	$a - b = 38°\ 12'$	$\alpha - \beta = 70°\ 2'$
$\beta = 50°\ 46.5'$	$\frac{1}{2}(a+b) = 81°\ 31'$	$\frac{1}{2}(\alpha+\beta) = 85°\ 47'$
or $129°\ 13.5'$	$\frac{1}{2}(a-b) = 19°\ 6'$	$\frac{1}{2}(\alpha-\beta) = 35°\ 1'$

Reject the larger value of β by the rule in I.

$\log \tan \frac{1}{2}(\alpha - \beta) = 9.8455$	$\log \tan \frac{1}{2}(a - b) = 9.5395$
$\log \sin \frac{1}{2}(a + b) = 9.9952$	$\log \sin \frac{1}{2}(\alpha + \beta) = 9.9989$
$\text{colog} \sin \frac{1}{2}(a - b) = 0.4852$	$\text{colog} \sin \frac{1}{2}(\alpha - \beta) = 0.2412$
$\log \cot \frac{1}{2}\gamma = 0.3259$	$\log \tan \frac{1}{2}c = 9.7796$
$\frac{1}{2}\gamma = 64°\ 43.5'$	$\frac{1}{2}c = 31°\ 3'$
$\gamma = 129°\ 27'$	$c = 62°\ 6'$

Check.	$\log \sin b = 9.9476$	$\log \sin \beta = 9.8891$
	$\sin c = \underline{9.9463}$	$\sin \gamma = \underline{9.8877}$
	0.0013	0.0014

NOTE. *In the solutions of triangles, a complete form should be prepared in advance,* so that only numerical values need be inserted when the tables are opened.

Case II. Given α, β, a. To find b, c, γ.

Formulas. $$\sin b = \frac{\sin \beta}{\sin \alpha} \sin a.$$

The rest of the calculations are as in Case I.

Case III. Given the three sides.

Example.

Given $a = 119° 32'$, $b = 44° 52'$, $c = 144° 50'$.

To find α, β, γ.

Formulas.

$$s = \frac{a+b+c}{2}; \quad \tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}};$$

$$\tan \frac{\alpha}{2} = \frac{\tan r}{\sin (s-a)}; \quad \tan \frac{\beta}{2} = \frac{\tan r}{\sin (s-b)}; \quad \tan \frac{\gamma}{2} = \frac{\tan r}{\sin (s-c)}.$$

Check. $$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

Computations.

$a = 119° 32'$	$\log \sin (s-a) = 9.7595$	$\frac{1}{2}\alpha = 38° 51.5'$
$b = 44° 52'$	$\log \sin (s-b) = 9.9737$	
$c = \underline{144° 50'}$	$\log \sin (s-c) = 9.2302$	$\frac{1}{2}\beta = 26° 12'$
$2s = 309° 14'$	$\text{colog } s = 0.3679$	
	$2 \, \lvert 9.3313$	$\frac{1}{2}\gamma = 69° 51.3'$
$s = 154° 37'$	$\log \tan r = 9.6657$	
$s - a = 35° 5'$		
$s - b = 109° 45'$	$\log \tan \frac{1}{2}\alpha = 9.9062$	$\alpha = 77° 43'$
$s - c = \underline{9° 47'}$	$\frac{1}{2}\beta = 9.6920$	$\beta = 52° 24'$
Check	$\frac{1}{2}\gamma = 0.4355$	$\gamma = 139° 43'$
sum. $153° 97'$		

	log	log	log
Check.	$\sin a = 9.9396$	$\sin b = 9.8485$	$\sin c = 9.7604$
	$\sin \alpha = \underline{9.9900}$	$\sin \beta = \underline{9.8989}$	$\sin \gamma = \underline{9.8108}$
difference:	9.9496	9.9496	9.9496

Case IV. Given α, β, γ. To find a, b, c.

Method (a). Solve the polar triangle as in Case III.

Method (b). Use formulas (12), (13), (17), (18), (19).
Check as in Case III.

Case V. Given two sides and their included angle.

Example.

Given $a = 103°\,7.0'$, $b = 70°\,40.0'$, $\gamma = 127°\,39.4'$.

To find α, β, c.

Formulas.

$$(23')\quad \tan \tfrac{1}{2}(\alpha+\beta) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}\gamma;$$

$$(22')\quad \tan \tfrac{1}{2}(\alpha-\beta) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}\gamma;$$

$$(1)\quad \sin c = \frac{\sin \gamma}{\sin \alpha} \sin a.$$

Check. (20)

$$\frac{\sin \frac{1}{2}(\alpha+\beta)}{\sin \frac{1}{2}(\alpha-\beta)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a-b)}$$

$$\begin{aligned} a &= 103°\ \ 7.0' \\ b &= \underline{\ 70°\,40.0'} \\ a+b &= 173°\,47.0' \\ a-b &= \ \ 32°\,27.0' \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(a+b) &= 86°\,53.5' \\ \tfrac{1}{2}(a-b) &= 16°\,13.5' \\ \tfrac{1}{2}\gamma &= 63°\,49.7' \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(\alpha+\beta) &= 83°\,26.7' \\ \tfrac{1}{2}(\alpha-\beta) &= \underline{\ \ 7°\,49.8'} \\ \alpha &= 91°\,16.5' \\ \beta &= 75°\,36.9'. \end{aligned}$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(a-b) &= 9.98235 \\ \operatorname{colog} \cos \tfrac{1}{2}(a+b) &= 1.26581 \\ \log \tan \tfrac{1}{2}\gamma &= \underline{9.69148} \\ \log \tan \tfrac{1}{2}(\alpha+\beta) &= 0.93964 \\ \log \sin \tfrac{1}{2}(a-b) &= 9.44624 \\ \operatorname{colog} \sin \tfrac{1}{2}(a+b) &= 0.00064 \\ \log \cot \tfrac{1}{2}\gamma &= \underline{9.69148} \\ \log \tan \tfrac{1}{2}(\alpha-\beta) &= 9.13836 \end{aligned}$$

$$\begin{aligned} \log \sin \gamma &= 9.89855 \\ \log \sin a &= 9.98852 \\ \operatorname{colog} \sin \alpha &= \underline{0.00011} \\ \log \sin c &= 9.88718 \end{aligned}$$

$$c = 180° - 50°\,27.8' = 129°\,32.2'.$$

$$\tfrac{1}{2}c = 64°\,46.1'.$$

Check.

$$\begin{aligned} \log \sin \tfrac{1}{2}(\alpha+\beta) &= 9.99715 \\ \log \sin \tfrac{1}{2}(\alpha-\beta) &= \underline{9.13429} \\ \text{difference:}\quad & 0.86286 \end{aligned}$$

$$\begin{aligned} \log \tan \tfrac{1}{2}c &= 0.32676 \\ \log \tan \tfrac{1}{2}(a-b) &= \underline{9.46390} \\ & 0.86286 \end{aligned}$$

Note that the quadrant of side c is determined by the fact that side c must be the longest side of the triangle.

Case VI. Given α, β, c.

Method (a). Solve the polar triangle by the method of Case V.

Method (b). Use (20′), (21′), (1); check by (22).

131. EXERCISES 50

Use 5-place tables where angles are given to fractions of minutes or to seconds. Reduce seconds to tenths of minutes.

1. $a = 70^\circ\ 5'$, $b = 63^\circ 22'$, $c = 59^\circ 17'$.

2. $a = 82^\circ 40'$, $b = 84^\circ 20'$, $c = 114^\circ 30'$.

3. $a = 150^\circ 20'$, $b = 137^\circ 20'$, $c = 20^\circ\ 6'$.

4. $a = 115^\circ 13.4'$, $b = 127^\circ 17.8'$, $c = 57^\circ 48.9'$.

5. $a = 54^\circ 40'$, $c = 131^\circ 30'$, $\beta = 96^\circ 47'$.

6. $a = 51^\circ 15'$, $b = 149^\circ 25'$, $\beta = 139^\circ 51'$.

7. $b = 112^\circ\ 0.3'$, $c = 95^\circ 13.3'$, $\alpha = 83^\circ 35.5'$.

8. $a = 63^\circ 51.5'$, $b = 144^\circ 13.4'$, $\gamma = 128^\circ 58.8'$.

9. $a = 132^\circ 39'$, $\beta = 52^\circ 38'$, $\gamma = 41^\circ 40'$.

10. $c = 51^\circ 44'$, $\alpha = 91^\circ 36'$, $\beta = 123^\circ 12'$.

11. $a = 123^\circ 43.8'$, $\beta = 127^\circ 41.8'$, $\gamma = 83^\circ 39.3'$.

12. $b = 47^\circ 42'$, $\alpha = 91^\circ 47.7'$, $\gamma = 55^\circ 52.7'$.

13. $\alpha = 55^\circ\ 7'$, $\beta = 148^\circ 41'$, $\gamma = 24^\circ 25'$.

14. $\alpha = 72^\circ 52'$, $\beta = 123^\circ 40'$, $\gamma = 101^\circ 45'$.

15. $\alpha = 108^\circ 45'$, $\beta = 140^\circ 50'$, $\gamma = 139^\circ 25'$.

16. $\alpha = 80^\circ 19.2'$, $\beta = 115^\circ 36.8'$, $\gamma = 79^\circ 10.5'$.

17. $b = 90^\circ 36'$, $c = 39^\circ 40'$, $\beta = 50^\circ 52'$.

18. $a = 114^\circ 27'$, $b = 84^\circ 22'$, $\beta = 80^\circ 19'$.

19. $a = 118^\circ 22'$, $b = 40^\circ\ 5.6'$, $\beta = 29^\circ 42.6'$.

20. $a = 143^\circ 39.7'$, $b = 133^\circ 29.3'$, $\alpha = 137^\circ 44.7'$.

21. $a = 137^\circ 30'$, $\alpha = 125^\circ\ 0'$, $\gamma = 41^\circ 50'$.

22. $a = 35^\circ 37.3'$, $\alpha = 29^\circ\ 3'$, $\beta = 45^\circ 44.1'$.

23. $a = 135^\circ 37.8'$, $\alpha = 129^\circ 14.7'$, $\beta = 110^\circ 47.3'$.

24. $a = 126^\circ 17.3'$, $\alpha = 117^\circ 44.6'$, $\gamma = 26^\circ 50.4'$.

25. $a = 69^\circ 10.0'$; $b = 31^\circ 35.2'$; $\gamma = 43^\circ 20.6'$.

26. $b = 125^\circ 59.3'$; $c = 170^\circ 10.9'$; $\alpha = 112^\circ 18.2'$.

27. $a = 18^\circ 48.7'$; $b = 159^\circ 20.5'$; $c = 158^\circ 14.3'$.

28. $a = 78^\circ 15.2'$; $b = 101^\circ 20.3'$; $\gamma = 111^\circ\ 3.7'$.

29. $a = 70^\circ\ 0'\ 37''$; $c = 63^\circ 47' 55''$; $\beta = 150^\circ 13' 15''$.

30. $c = 125^\circ 18' 20''$; $\alpha = 96^\circ\ 2' 12''$; $\beta = 102^\circ 16' 36''$.

31. $c = 120^\circ 18' 33''$; $\alpha = 27^\circ 22' 34''$; $\beta = 91^\circ 26' 44''$.

32. $\alpha = 153^\circ 17'\ 6''$; $\beta = 78^\circ 43' 32''$; $\gamma = 78^\circ 15' 46''$.

33. $\alpha = 112^\circ 10' 40''$; $\beta = 67^\circ 49' 30''$; $\gamma = 43^\circ\ 1'\ 0''$.

34. $a = 80^\circ 34' 20''$; $\beta = 132^\circ 26' 10''$; $\gamma = 52^\circ 28' 15''$.

35. $b = 159^\circ 20.5'$; $c = 158^\circ 14.3'$; $\gamma = 112^\circ 14.2'$.

36. $a = 165^\circ 25' 20''$; $\alpha = 112^\circ 10' 40''$; $\beta = 67^\circ 49' 30''$.

37. $a = 23^\circ 57' 11''$; $c = 120^\circ 18' 33''$; $\gamma = 102^\circ\ 5' 46''$.

38. $\alpha = 58^\circ 12.7'$; $\gamma = 169^\circ 18.2'$; $c = 170^\circ 10.9'$.

39. $b = 88^\circ 12' 19''$; $c = 86^\circ 15' 15''$; $\beta = 78^\circ 43' 32''$.

40. $a = 80^\circ 34' 20''$; $\alpha = 49^\circ 32' 25''$; $\gamma = 52^\circ 28' 15''$.

CHAPTER

XII

APPLICATIONS

132. Terrestrial triangles.

We shall consider the earth as a sphere with a radius of 3960 statute miles, or land miles. Longitudes are to be reckoned from Greenwich as prime meridian, 180° or 12 hours to the west or east. The direction will be indicated by a letter, W or E; when signs are used, + means west longitude.

We shall denote longitude by lambda, λ. Then the longitude of a given place is measured by the arc of the equator contained between the meridian of Greenwich and the meridian of the place, and it is also measured by the angle at the pole between those two meridians.

We shall denote latitude by the letter phi, φ or by L. Latitude is counted positive to the north, and negative to the south, of the equator.

We shall denote distance from the north pole by p. This polar distance will be the complement of the latitude,

$$p = 90° - \varphi = 90° - \text{L}.$$

A triangle whose vertices are the north pole (or the south pole) and two points on the earth's surface will be called a terrestrial triangle.

In Fig. 81, let P be the earth's north pole, G Greenwich, A and A' two stations, station A' lying to the west of station A. Then triangle APA' is a *terrestrial triangle.* Two sides of this triangle are the polar distances of the two stations, or the complements of their latitudes, and the third side is the great circle arc between the two stations. The angle at the pole is the difference of longitude of the two stations. The other two angles are the angles which the great circle arc AA' makes with the meridian at the respective stations.

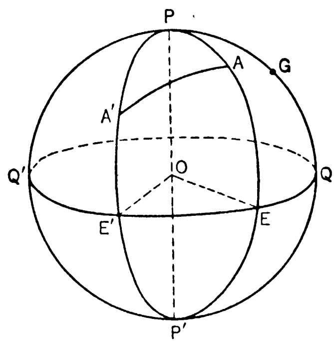


FIG. 81

To sail a ship, or fly an airplane, from A to A' the navigator would wish to know the length of the journey if the great circle arc AA' were followed, and the angles PAA' and $PA'A$, which would be the courses of departure from A and of arrival at A'. The problem comes under Case V of §127, or the alternative method of §128, or the haversine method of §129.

The nautical mile is defined as the length of an arc of $1'$ of a great circle on the earth's surface. Accordingly, the circumference of the earth would be $360 \times 60 = 21600$ nautical miles. The circumference in statute or geographic miles is 24890 miles. Roughly speaking, the measure of a distance in geographic miles is about one-seventh greater than the measure of the same distance in nautical miles.

From the definition of the nautical mile, it follows that the number of minutes in an arc of a great circle is also the number of nautical miles in that arc.

133. Problems involving the terrestrial triangle. Great circle sailing.

Problem 1.

What is the great circle distance from Seattle, (47° 40′ N, 122° 20′ W) to Honolulu, (21° 20′ N, 157° 50′ W)?

We shall use the method of §128 with A as the north pole, B as the point in Seattle whose latitude and longitude are as given above, C the corresponding point in Honolulu; we shall have

$c = AB = 90° - 47° 40' = 42° 20'$;
$b = AC = 90° - 21° 20' = 68° 40'$;
$\alpha = \text{angle } BAC = \text{diff. of long.} = 157° 50' - 122° 20' = 35° 30'$.

With these values we calculate m from (28) and then a from (29) of §128.

Computations.

$$
\begin{array}{rlrl}
\tan m &= \tan b \cos \alpha; & \cos a &= \cos b \sec m \cos (c - m) \\
\log \tan b &= 0.4083 & \log \cos b &= 9.5609 \\
\log \cos \alpha &= \underline{9.9107} & \text{colog} \cos m &= 0.3639 \\
\log \tan m &= 0.3190 & \log \cos (c - m) &= \underline{9.9671} \\
 & & \log \cos a &= 9.8919 \\
m &= 64° 22' & & \\
c - m &= 22° \ \ 2' & a &= 38° 46'.
\end{array}
$$

$$a = 38 \times 60 + 46 = 2326 \text{ nautical miles.}$$

Check. Use the half-angle formula, squared,

$$\sin^2 \frac{\alpha}{2} = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}.$$

$$
\begin{array}{rrrrlrrr}
 & ° & ' & & & & & \\
a = & 38 & 46 & \log \sin (s-b) = & 9.0345 & & \tfrac{1}{2}\alpha = & 17° 45' \\
b = & 68 & 40 & \log \sin (s-c) = & 9.7308 & & & \\
c = & \underline{42} & \underline{20} & \text{colog} \sin b = & 0.0308 & & \log \sin \tfrac{1}{2}\alpha = & 9.4841 \\
2s = & 148 & 106 & \text{colog} \sin c = & \underline{0.1717} & & & \underline{2} \\
 & & & \text{sum} & 18.9680 & & & 18.9682 \\
s = & 74 & 53 & & & & & \\
s - b = & 6 & 13 & & & & & \\
s - c = & 32 & 33 & & & & &
\end{array}
$$

A check could also be made by calculating both m and n and then using both forms of (29), §128.

Haversine solution. From (34) of §129, we have

$$\text{hav}\ a = \text{hav}\ (b - c) + \sin b \sin c\ \text{hav}\ \alpha (= \text{hav}\ (b - c) + Z).$$

$$\begin{array}{rlrlrl}
\alpha &= 35° 30' & \log \sin b &= 9.9692 & \text{hav}\ (b - c) &= 0.0519 \\
b &= 68° 40' & \log \sin c &= 9.8283 & Z &= \underline{0.0583} \\
c &= \underline{42° 20'} & \log \text{hav}\ \alpha &= \underline{8.9682} & \text{hav}\ a &= 0.1102 \\
b - c &= 26° 20' & \log Z &= 8.7657 & a &= 38° 47'
\end{array}$$

Problem 2.

With the data of Problem 1, calculate the course of the ship on leaving Seattle and on arriving at Honolulu.

NOTE. Here the term "*course*" means the angle between the direction in which the ship is headed and the meridian. The angle is measured from the northern or southern part of the meridian to make the course an acute angle. It corresponds to the surveyor's use of the term "bearing" (§56). The navigator uses the term bearing as an angle measured from the direction of the keel of his ship.

We have to calculate angles β and γ. We use (28), (30), (31) of §128.

$$\begin{array}{ll}
\tan m = \tan b \cos \alpha; & \cot \beta = \cot \alpha \csc m \sin (c - m). \\
\tan n = \tan c \cos \alpha; & \cot \gamma = \cot \alpha \csc n \sin (b - n).
\end{array}$$

$$\begin{array}{rlrlrl}
 & \ \ \circ\ \ \ ' & & & & \\
\alpha &= 35\ 30 & \log \tan b &= 0.4083 & \log \cot \alpha &= 0.1467 \\
b &= 68\ 40 & \log \cos \alpha &= \underline{9.9107} & \text{colog} \sin m &= 0.0450 \\
c &= 42\ 20 & \log \tan m &= 0.3190 & \log \sin (c - m) &= \underline{9.5742n} \\
b - c &= 26\ 20 & m &= 64° 22' & \log \cot \beta &= 9.7659n \\
 & & c - m &= -22° 2' & \beta = 180° - 59° 45' &= 120° 15'.
\end{array}$$

Course: S 59° 45′ W

$$\begin{array}{rlrl}
\log \tan c &= 9.9595 & \log \cot \alpha &= 0.1467 \\
\log \cos \alpha &= \underline{9.9107} & \text{colog} \sin n &= 0.2253 \\
\log \tan n &= 9.8702 & \log \sin (b - n) &= \underline{9.7258} \\
n &= 36° 32' & \log \cot \gamma &= 0.0978 \\
b - n &= 32°\ 8' & \gamma &= 38° 38'
\end{array}$$

Course: S 38° 38′ W.

Check. $\dfrac{\sin b}{\sin c} = \dfrac{\sin \beta}{\sin \gamma}.$

$$\begin{array}{lrlrl}
 & \log \sin b &= 9.9692 & \log \sin \beta &= 9.9364 \\
 & \sin c &= \underline{9.8283} & \sin \gamma &= \underline{9.7954} \\
\text{diff.} & & 0.1409 & & 0.1410
\end{array}$$

Problem 3.

If the ship in Problem 1 follows the great circle track Seattle-Honolulu, what should be her course when 1000 nautical miles out of Seattle? What will be her latitude and longitude at that point?

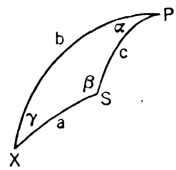


FIG. 90

Known values are:

$c = 42° \, 20'$,
$a = 1000' = 16° \, 40'$,
$\beta = 120° \, 15'$. (Prob. 2.)

We calculate b, α, γ, from which the required quantities can be obtained.

To obtain b we use (34) of §129 with proper change of letters. Then α and γ are obtained by the law of sines.

$$\text{hav } b = \text{hav } (c - a) + \sin a \sin c \text{ hav } \beta \; (= \text{hav } (c - a) + Z);$$

$$\sin \alpha = \frac{\sin a}{\sin b} \sin \beta; \quad \sin \gamma = \frac{\sin c}{\sin b} \sin \beta.$$

Check. Use (20′) of §124, with change of letters and cleared of fractions. $\tan \frac{1}{2}(c - a) \sin \frac{1}{2}(\gamma + \alpha) = \sin \frac{1}{2}(\gamma - \alpha) \tan \frac{1}{2}b$.

Computations.

	° ′	log		
β =	120 15	$\sin a = 9.4576$	$\text{hav } (c - a) =$	0.0493
c =	42 20	$\sin c = 9.8283$	$Z =$	0.1452
a =	16 40	$\text{hav } \beta = 9.8761$	$\text{hav } b =$	0.1945
$c - a$ =	25 40	$\log Z = 9.1620$	$b =$	$52° \, 20'$
$\frac{1}{2}(c - a)$ =	12 50		$\frac{1}{2} =$	$26° \, 10'$

		log *Check.*	
$\log \sin a = 9.4576$	$\log \sin c = 9.8283$	$\tan \frac{1}{2}(c - a) =$	9.3576
$\log \sin \beta = 9.9365$	$\log \sin \beta = 9.9365$	$\sin \frac{1}{2}(\gamma + \alpha) =$	9.7334
$\text{cl} \sin b = 0.1015$	$\text{cl} \sin b = 0.1015$		
$\log \sin \alpha = 9.4956$	$\log \sin \gamma = 9.8663$	sum	9.0910
	$\gamma = 47° \, 18'$		
	$\alpha = 18° \, 14'$	$\sin \frac{1}{2}(\gamma - \alpha) =$	9.3996
		$\tan \frac{1}{2}b =$	9.6914
$\gamma + \alpha = 65° \, 32'$	$\frac{1}{2}(\gamma + \alpha) = 32° \, 46'$	sum	9.0910
$\gamma - \alpha = 29° \;\; 4'$	$\frac{1}{2}(\gamma - \alpha) = 14° \, 32'$		

Ans. At X, 1000 miles from Seattle on g.c. SH,
latitude $= 90° - b = 37° 40'$,
longitude $= \alpha +$ long. of S $= 18° 14' + 122° 20' = 140° 34'$,
course = S 47° 18′ W.

EXERCISES 51

1. Calculate the latitude, longitude and course on g.c. Seattle-Honolulu when the distance from Seattle is:

(a) 500 miles; (b) 1500 miles; (c) 2000 miles.

2. Calculate the latitude and course on g.c. Seattle-Honolulu, and the distance out from Seattle at intervals of 10° in longitude from Seattle.

Suggestion. In triangle SPX, Fig. 90, we now have side SP, angle β as found in Problem 2, and angle $SPX = 10°$ for the first interval. The solution comes under Case VI of §127.

3. In what longitude does the great circle from Seattle to Honolulu cross the 40th parallel of latitude? What is the great circle distance from Seattle at this point?

4. An airplane pilot, flying from Seattle to Honolulu, finds that his position is 30° N, 150° W. How far should he now fly, directly north or south, to get on the great circle?

5. Use the methods of §127, §128, and §129, to calculate the great circle distance between San Francisco (37° 47′ N, 122° 26′ W) and Melbourne (37° 50′ S, 145° 0′ E).

6. Determine the positions of the "vertices" of the great circle through San Francisco and Melbourne. Which one would be used in the vertex method of determining positions on the great circle?

7. Determine the longitude of the point at which the great circle from San Francisco to Melbourne crosses the equator. What is the great circle course at that point? What is the distance from San Francisco?

134. Great circle positions and courses. Vertex method.

In deriving the fundamental formulas relating to the spherical oblique triangle (§120 and §128) we used as a basis for the proofs the right triangles formed by drawing an arc through one of the vertices and perpendicular to the opposite side. We follow this plan now. As before, two cases will arise according as the perpendicular falls within the base or on the base produced.

NOTE. In §128 the unknown parts of the triangle were expressed in terms of the given parts and an auxiliary arc m. In the present section both m and p (or their equivalents e and p) are used to determine the unknown parts. Here e is taken as positive.

In Fig. 91 we have a g.c. $CBVV'$ part of which, namely arc BC, is assumed to be the track of a ship sailing from B to C.

In §133 we calculated the distance BC and the courses at B and C. We also calculated the position (latitude and longitude) of a point X, lying on BC at a given distance from B, and the course at X.

When several points like X are chosen to break a long arc into smaller segments, a convenient method for calculating the positions and courses at these points will now be explained. We call it the " Vertex Method."

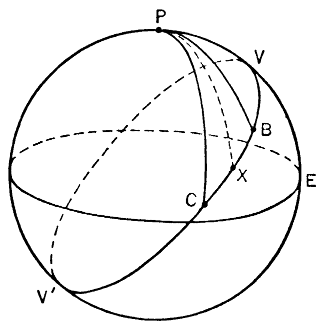


FIG. 91

Follow around on the great circle to the point where it is farthest from the equator. Call this point V, the " vertex " of g.c. BC. There is of course the opposite point V' where the g.c. is again farthest from the equator. In a given case there will be no question about which one to use.

At V the g.c. BC cuts the meridian EP at right angles. So arc PV is the perpendicular drawn from P on BC produced. To find the position and course at X we use right triangle PVX.

If point B were taken to lie beyond vertex V, the foot of the perpendicular, or point V, would fall *between* B and C.

We represent the two cases in the figures which follow. Angle β is acute in the first figure, obtuse in the second.

In either figure, P is the north pole, B and C are two points of known latitude and longitude, V is the northern vertex of

g.c. BC (that is, the foot of the meridian arc PV drawn perpendicular to BC), X is a point on the g.c. track at a given distance from B.

To determine the latitude, longitude, and course at X we first solve right triangle PVB, then right triangle PVX.

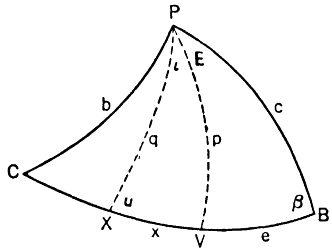


Fig. 92a

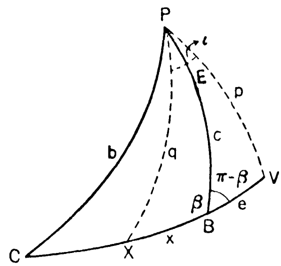


Fig. 92b

We first calculate angle β, from the known positions of B and C.

Then, in right triangle PVB we have the angle at B and the hypotenuse c.

Napier's rules give the formulas for p, e, E:

1) $\sin p = \sin c \sin \beta,$ *Check.*

2) $\tan e = \pm \tan c \cos \beta,$ $\sin p = \tan e \cot E.$

3) $\cot E = \pm \cos c \tan \beta.$

Use the $+$ sign if $\beta < 90°$, the $-$ sign if $\beta > 90°$.

In right triangle PVX we now have p, the polar distance of V, and arc $VX = x = BX \mp e$, $\mp$ according as $\beta \lessgtr 90°$.

Napier's rules give the formulas for q, l, u:

4) $\cos q = \cos p \cos x,$ *Check.*

5) $\cot l = \sin p \cot x,$ $\cos q = \cot l \cot u.$

6) $\cot u = \cot p \sin x.$

The coordinates of V are: $\varphi_V = 90° - p$; $\lambda_V = \lambda_B \pm E.$

The coordinates of X are: $\varphi_X = 90° - q$; $\lambda_X = \lambda_V + l.$

Course at $X = u.$

It may be noted that we might consider the values of e and E as signed numbers and drop the ambiguous sign $\pm$ in formulas 2) and 3) and in following equations.

Example.

We use S and H of Problem 1, §133, as B and C respectively. Also we take point X so that $BX = 1000$ miles $= 1000' = 16° 40'$. We keep the letters used in equations 1) to 6).

	From given data	Computed	
In PSH:	$b = 68° 40'$	$a = 38° 46'$	(Prob. 1)
	$c = 42° 20'$	$\beta = 120° 15'$	(Prob. 2)
	$\alpha = 35° 30'$	$\gamma = 38° 38'$	(Prob. 2)

Since $\beta > 90°$ we use the lower signs.

Calculation of p, e, E, with $c = 42° 20'$ and $\beta = 120° 15'$.

log	log	log
$\sin c = 9.8283$	$\tan c = 9.9595$	$\cos c = 9.8688$
$\sin \beta = \underline{9.9365}$	$\cos \beta = \underline{9.7022n}$	$\tan \beta = \underline{0.2341}$
$\sin p = 9.7648$	$\tan e = 9.6617$	$\cot E = 0.1029$
$p = 35° 35'$	$e = 24° 39'$	$E = 38° 16'$

$$x = BX + e = 16° 40' + 24° 39' = 41° 19'.$$

Calculation of q, l, u; with $p = 35° 35'$ and $x = 41° 19'$.

log	log	log
$\cos p = 9.9102$	$\sin p = 9.7648$	$\cot p = 0.1454$
$\cos x = \underline{9.8757}$	$\cot x = \underline{0.0560}$	$\sin x = \underline{9.8197}$
$\cos q = 9.7859$	$\cot l = 9.8208$	$\cot u = 9.9651$
$q = 52° 21'$	$l = 56° 30'$	$u = 47° 18'$

Coordinates of V:	$\varphi = 54° 25'$,	$\lambda = 84° 04'$.
Coordinates of X:	$\varphi = 37° 39'$,	$\lambda = 140° 34'$.
Course at X:	$u = 47° 18'$.	

The student is advised to solve this example independently, without using the special notation of formulas 1) . . . 6). Napier's Rules are sufficient. Two steps are involved:

(a) solve right triangle PSV, given side PS and angle PSV;
(b) solve right triangle PXV, given side PV and angle SPV.

EXERCISES

1. Solve Exercise 1 of §133 by the vertex method.

2. Solve Exercise 2 of §133 by the vertex method.

3. If the signs $\pm$ in formula 2) are dropped, and the equation is written $\tan e = \tan c \cos \beta$, examine the signs of side e according to the quadrants of side c and angle β. Similarly for angle E in formula 3).

135. Terrestrial coordinates of selected stations.

This list is placed here to afford material for drill exercises.

Place	Lat. ° ′	Long. ° ′	Place	Lat. ° ′	Long. ° ′
Berlin	+ 52 30	− 13 22	Montreal	+ 45 30	+ 73 35
Bombay	+ 18 54	− 72 49	Moscow	+ 55 45	− 37 34
Boston	+ 42 22	+ 71 4	New York	+ 40 45	+ 73 56
Cape of Good Hope	− 33 21	− 18 30	Paris	+ 48 50	− 2 20
Dutch Harbor	+ 53 53	+ 166 35	Rio de Janeiro	− 22 54	+ 43 10
Greenwich	+ 51 29	0 0	Rome	+ 41 54	− 12 29
Havana	+ 23 10	+ 82 22	San Francisco	+ 37 47	+ 122 26
Hong Kong	+ 22 18	− 114 10	San Luis	− 33 18	+ 66 20
Honolulu	+ 21 20	+ 157 50	Santiago	− 33 34	+ 70 41
Johannesburg	− 26 11	− 28 4	Seattle	+ 47 40	+ 122 20
Leningrad	+ 59 56	− 30 17	Singapore	+ 1 18	− 103 51
Liverpool	+ 53 24	+ 3 4	Sydney	− 33 52	− 151 12
Manila	+ 14 35	− 120 59	Tokyo	+ 35 39	− 139 45
Mare Island	+ 38 6	+ 122 16	Valparaiso	− 33 2	+ 71 39
Melbourne	− 37 50	− 144 59	Washington	+ 38 55	+ 77 4
Mexico City	+ 19 26	+ 99 7	Wellington	− 41 8	− 174 46

136. EXERCISES 52

1. Calculate the sides (in statute miles), the angles, and the area (in square miles) of the triangle whose vertices are:

New York — San Francisco — Mexico City.

2. Calculate the sides (in nautical miles), the angles, and the area (in square miles) of the triangle whose vertices are:

New York — Rio de Janeiro — Liverpool.

3. Find the distance along the great circle from Boston to Wellington in New Zealand.

4. A vessel sails on a great circle from San Francisco to Sydney. Find the courses of departure and arrival and the distance sailed.

5. If the vessel in Exercise 4 is on the great circle 1440 nautical miles out from San Francisco, what is her position (φ and λ) and on what course is she sailing?

6. An airplane is to fly from Dutch Harbor to Tokyo. Calculate the great circle distance and courses of departure and arrival.

7. As in Exercise 6, for a flight from Manila to Tokyo.

8. (a) Calculate the great circle distance, Sydney-Valparaiso. (b) Calculate φ and λ for points on this great circle at intervals of 10° from Sydney.

9. Find the shortest distance between two points on the Arctic circle which differ by four hours in longitude. How far is it between these points on the Arctic circle?

10. If a person were to start from a point in 80° north latitude and go always directly east for a distance of 2000 miles, how much shorter would the great circle distance be?

137. Rhumb line. Mercator chart.

Rhumb line.

Any great circle track, except the equator or the meridians, will cut successive meridians at a constantly changing angle. In the problems of §133 we saw that the great circle Seattle-Honolulu cuts the meridian through Seattle at an angle 59° 45′; 1000 miles from Seattle on the great circle the angle is 47° 18′; at Honolulu the angle is 38° 38′. Therefore, to follow the g.c. track, the navigator would have to change continually the course of his ship.

To avoid this impossible performance, the latitudes and longitudes of a number of points on the great circle are calculated, and the ship proceeds from one point to the next by following a track which is not a great circle but which cuts all meridians at the same angle and is called a *rhumb line.* This line is longer than the g.c. track. But for moderate distances the difference of length is small, and is more than offset by the convenience of steering a fixed course.

The problem arises: What course must be set to go from a given point A to a second given point B, without changing the course? This problem is solved by use of the Mercator chart.

On such a chart the track of a ship or airplane which travels on a fixed course appears as a straight line, the rhumb line. Meridians appear on the chart as parallel straight lines, all of which are cut at the same angle by the rhumb line. A graphic solution of the problem is, therefore, obtained by marking the positions of the two joints A and B on a Mercator Chart, joining them by a straight line, and measuring the angle at which this line cuts any meridian.

The Mercator chart.

The theory of this chart can not be discussed here. We shall only indicate the plan of its construction and how it is used.

Imagine a cylinder to be wrapped around the earth touching the earth's surface along the equator, the axis of the cylinder coinciding with the earth's polar axis extended in both directions.

To make a map of the earth's surface on this cylindrical sheet, we obtain the point S on the cylinder which corresponds to a point R on the earth's surface by constructing the broken line ORS as shown in Fig. 93. If the radius were continued directly on it would meet the cylinder in a point higher up, and a small increase in the latitude of the point R on the surface of the earth would lead to a great increase in the height of the corresponding point on the cylinder. To moderate somewhat this rapid in-

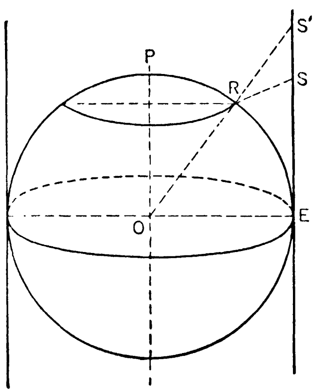


FIG. 93

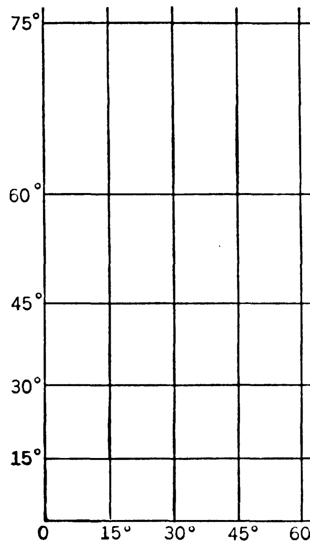


FIG. 94

crease in height of the point S' as the point R moves toward the pole, line RS is drawn at an angle to OR which is determined by the theory of the map.

By this construction every point R on the surface of the earth will lead to a point S on the cylinder.

If the point R follows a meridian as ERP, the point S will move up on the cylinder following a straight line which is an element of the cylinder. If we draw meridians on the earth's surface, say at intervals of 15° of longitude, we can imagine the corresponding straight lines drawn on the cylinder. These will be elements of the cylinder spaced equally around the cylinder.

If the point R describes a parallel of latitude, the point S will move around the cylinder in a circle parallel to the equator and at a distance ES above the equator. If we draw several parallels of latitude, say 15° apart, they will lead to circles on the cylinder with unequal spacing, the spaces becoming wider as we go north.

If we now cut the cylinder open along one of its elements and roll it out flat, we will have a plane map on which the meridians of the earth's surface are represented by parallel straight lines which are elements of the original cylinder. Equally spaced meridians will correspond to equally spaced parallel lines. Each parallel of latitude will be represented on the plane map by a straight line parallel to the line which represents the equator. As the latitude parallels are taken farther north the spacing between the corresponding lines on the map will increase rapidly as we approach the pole. (Fig. 94.)

If two points, A and B, are selected on the surface of the earth, and if they are "projected" on the surface of the cylinder to yield the points A' and B', these points will then appear on our plane map. The great circle track AB could be represented point by point and would yield a curve on the map.

The plane map which we obtain in this manner is called a *Mercator chart.* On this chart a great circle arc AB will appear as a curve joining the corresponding points A' and B'. The straight line on the map joining points A' and B' is the *rhumb line.* The angle which this line makes with the meridians will show the navigator the fixed course to sail from A to B on the earth's surface.

The rapid increase of distance between the parallels of latitude on the map which correspond to equally spaced parallels of latitude on the earth's surface causes distortion. As is seen by inspection, a 15° change of latitude on the earth requires a wider spacing of the parallels on the map as we move north. There is also distortion due to the fact that on the earth's surface two meridians converge as we approach the pole, but on the map the lines representing these meridians are par-

allel. Because of this distortion or stretching lines on the map must be reduced to obtain their equivalent length on the earth's surface. The theory of the map tells us that a short line segment on the map must be multiplied by the cosine of the latitude to obtain the corresponding length on the earth's surface. If the two ends of the line segment lie in different latitudes, we multiply by the *cosine of the middle latitude* as a close approximation.

138. Construction of a Mercator chart.

Let point A be situated on the earth's surface in latitude φ and longitude λ. Let point A', which represents A on the map, be placed at a distance x from the meridian of Greenwich and at a distance y from the equator. We shall call x and y the *Mercator coordinates* of point A'. They are calculated by the following formulas, as multiples of the unit which is used to represent $1'$ of longitude on the equator.

(1) $\qquad x = \lambda$ (in minutes); $\quad y = 7915.71 \log \cot \tfrac{1}{2}p.$

Here p is the polar distance of point A. The numerical factor in the value of y is given more accurately than is needed for our calculations; its logarithm to seven places is 3.8984896.

Fig. 95 represents a Mercator projection of a portion of the earth's surface, including the arc from Seattle to Honolulu.

Example.

	Longitude	Latitude	x	y
S = Seattle	$122°\ 20' = 7340'$	$47°\ 40'$	7340	3261.8
H = Honolulu	$157°\ 50' = 9470'$	$21°\ 20'$	9470	1310.7

To make a chart showing the points (x, y) which represent S and H respectively we choose a suitable scale along the equator, say 1 inch = 7.5° = 450′ of longitude. The points S' and H' are the opposite corners of a rectangle whose width is the difference of the two values of x, $9470 - 7340 = 2130$; its height is the difference of the two values of y, $3261.8 - 1310.7 = 1951.1$. On the indicated scale the rectangle would be about $4\tfrac{1}{2}$ inches wide and $4\tfrac{1}{4}$ inches high.

For any two stations A and B of given latitudes and longitudes we can calculate the coordinate (x, y) and construct a rectangle with A and B at opposite corners.

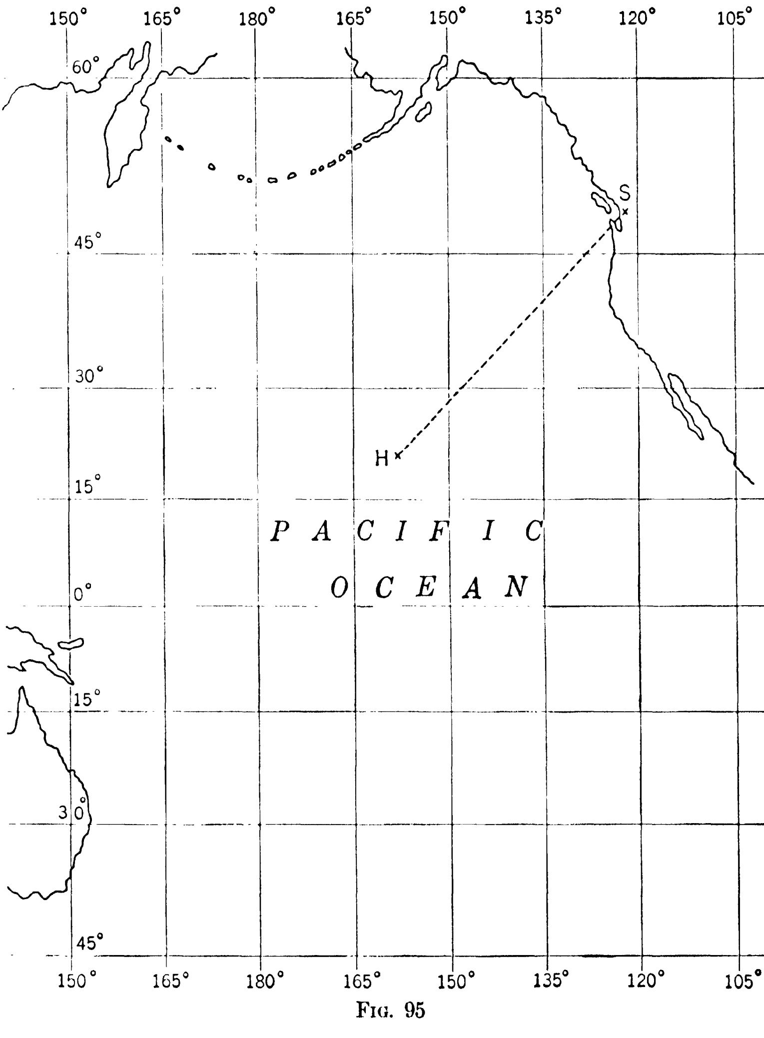


FIG. 95

In calculations involving latitude the *middle latitude* of the rectangle is used, as stated at the end of §137.

Notation. Let

D = the length of the diagonal of the rectangle;
$\triangle x$ = the difference of the x-values or width of rectangle;
$\triangle y$ = the difference of the y-values or height of rectangle;
C = the acute angle between the diagonal and a meridian.

Problem 1.

To find the rhumb line course (r.l.C) and the rhumb line distance (r.l.d) between two points of given latitudes and longitudes.

Angle C above defined is the rhumb line course. Therefore

$$\tan \text{r.l.C} = \frac{\Delta x}{\Delta y}. \tag{2}$$

The r.l. distance can be calculated, though only approximately, by multiplying D by the cosine of the middle latitude, $\cos \varphi_m$, to get the corresponding length on the earth's surface, as stated at the end of §137. But $D = \triangle y \sec C$ and therefore

$$\text{r.l.d} = \triangle y \sec C \cos \varphi_m, \text{ approximately.}$$

Now by reducing $\triangle y$ by the factor $\cos \varphi_m$ it becomes $\triangle \varphi$, the difference of latitude in minutes: $\triangle y \cos \varphi_m = \triangle \varphi$, approximately. Substituting this in the preceding equation we have

$$\text{r.l.d} = \triangle \varphi \sec (\text{r.l.C}). \tag{3}$$

Example.

Determine the r.l. course and the r.l. distance between $A(40° \text{ N}, 40° \text{ W})$ and $B(43° \text{ N}, 43° \text{ W})$.

From (1), for A, $x = 2400$, $y = 2624$;
for B, $x = 2580$, $y = 2863$.

$\triangle x = 180$, $\triangle y = 239$, $\tan (\text{r.l.C}) = \frac{180}{239}$, $\text{r.l.C} = \text{N } 36° 59' \text{ W}$.

$\text{r.l.d} = \triangle \varphi \sec (\text{r.l.C}) = 180 \sec 36° 59' = 225.3'$.

Problem 2.

A ship starts from point A of given latitude and longitude and steams a distance d at a fixed course angle C; to determine the change in latitude and in longitude.

For the change in latitude equation (3) gives

$$\triangle \varphi = \text{r.l.d} \cos (\text{r.l.C}). \tag{4}$$

For the change in longitude, $\triangle \lambda$, we have

(5) $\triangle \lambda = \triangle x \sec \varphi_m = d \sin C \sec \varphi_m$, approximately.

We may note here that $\triangle x$ is the same as "departure" in plane surveying or plane sailing and lies along a parallel of latitude. The length of the corresponding segment of the equator, or the difference of longitude, is $\triangle x \sec \varphi_m$.

Example.

A ship starts from (40° N, 40° W) and steams 225 miles on course N 37° W. Determine the latitude and longitude arrived at.

$$\begin{aligned}
\triangle \varphi &= 225 \cos 37° = 179.7' = 2° \, 59.7', \\
\varphi &= 40° + 2° \, 59.7' = 42° \, 59.7'. \\
\triangle \lambda &= 225 \sin 37° \sec 41° \, 30' = 180.8' = 3° \, 0.8' \\
\lambda &= 40° + 3° \, 0.8' = 43° \, 0.8'.
\end{aligned}$$

Note that the values of d and C here given are practically the values calculated in the preceding example.

EXERCISES 53

1. Calculate the values of r.l.C and of r.l.d for the rhumb line track A(40° N, 43° W) to B(43° N, 40° W).

2. (a) Construct the framework (grid) of a Mercator chart for $\varphi = 15°$, 30°, 45°, 60° and $\lambda = 120°$, 135°, 150°, 165°. (b) Calculate the Mercator coordinates of Seattle and Honolulu and mark them on the chart.

3. (a) Construct a Mercator grid for the region $\lambda = 165°$ E eastward to $\lambda = 75°$ W, and $\varphi = 30°$ S to 60° S. (b) Calculate the Mercator coordinates of Sydney and Valparaiso and mark them on the map. (c) Plot on this chart the positions of the great circle points calculated in Exercise 8(b) of §136.

4. Determine the g.c. distance and the rhumb line distance from New York to Boston.

5. Determine the latitude in which the rhumb line Seattle-Honolulu cuts the 135th meridian.

6. Compare the rhumb line distance between two points on the arctic circle which are separated by 180° longitude with the great circle distance between these points.

7. Two stations both in the northern hemisphere are separated by 5° in latitude and 30° in longitude. What can you say about the position of the great circle between these stations, whether it is north or south of the rhumb line?

8. An airplane is flying on the great circle track from Seattle to Honolulu at 200 knots per hour. What are the coordinates of the position reached when four hours out of Seattle?

139. Summary of methods used in navigation.

We first note that there are three ways of reckoning the course angle C.

1) The acute angle between heading of ship and the meridian; call this C_1.

2) The angle between heading of ship and meridian counted from the north (or south in the southern hemisphere) through the east or west from 0° to 180°. Call this C_2.

3) The angle between the heading of ship and the north (or south in the southern hemisphere) through the east, from 0° to 360°. Call this C_3.

Plane Sailing. §56.

$$\text{diff. lat.} = d \cos C; \quad \text{departure} = d \sin C.$$

Here C is C_1 or C_2; if C_2, diff. lat. is a signed number.

Traverse Sailing.

Plane sailing when the track consists of several legs, as in Exercise 1 of §56.

Parallel Sailing.

The course is due east or west, along a parallel of latitude.

$$\text{diff. lat.} = 0,\ \text{departure} = d,\ \text{the distance run.}$$
$$\text{diff. long.} = \text{departure times sec } \varphi.$$

Middle Latitude Sailing.

$$\text{diff. lat.} = d \cos C; \quad \text{departure} = d \sin C.$$
$$\text{diff. long.} = \text{departure times sec } \varphi_m,\ \text{approximately.}$$

Here the use of the middle latitude φ_m takes account, at least approximately, of the convergence of the meridians. The two stations must lie on the same side of the equator.

Great Circle Sailing. §133, §134.

I. To find the g.c. distance and the initial g.c. course for the g.c. track from A to B.

(a) Solve triangle PAB as under Case V of §127; or
(b) use the haversine formulas:

$$\text{hav } d = \text{hav } |\Delta\varphi| + \cos\varphi_B \cos\varphi_A \text{ hav } |\Delta\lambda|. \quad \text{§129, (34).}$$
$$\sin C = \cos\varphi_B \sin\Delta\lambda \csc d. \quad \text{(Law of sines.)}$$

When the quadrant of C is not known in advance, calculate C from

$$\text{hav } C = \frac{\text{hav co}\,\varphi_B - \text{hav}\,|d - \text{co}\,\varphi_A|}{\sin d \sin \text{co}\,\varphi_A}. \quad \text{§129, (33).}$$

II. Coordinates of vertex of g.c. track.

$$\cos\varphi_V = \cos\varphi_A \sin C; \quad \tan(\lambda_V - \lambda_A) = \csc\varphi_A \cot C.$$

III. Latitude of point X on g.c. track AB when the longitude of X is given.

$$\cot\varphi_X = \cot\varphi_V \sec|\lambda_X - \lambda_V|.$$

Composite Sailing.

A combination of g.c. sailing and parallel sailing when the g.c. track reaches too high altitudes. A selected part of the " top " of the g.c. is cut off by a parallel of latitude.

Mercator Sailing. §138.

I. To determine the rhumb line course and the rhumb line distance between two given points:

$$\tan \text{r.l.C} = \frac{\Delta x}{\Delta y}; \quad \text{r.l.d} = \Delta\varphi \sec(\text{r.l.C}).$$
$$x = \lambda \text{ (in minutes)}; \quad y = 7915.71 \log\cot \tfrac{1}{2}p.$$

II. To determine the change in latitude and longitude due to sailing a given r.l. course and distance:

$$\Delta\varphi = \text{r.l.d} \cos(\text{r.l.C}); \quad \Delta\lambda = d \sin c \sec\varphi_m, \text{ approximately.}$$

140. Applications to the celestial sphere.

For the purpose of this article we assume the *celestial sphere* to be an indefinitely large sphere concentric with that of the earth. On it as a background we see all celestial objects.

The projections on the celestial sphere of the earth's poles, equator, meridians and parallels of latitude are named respec-

tively the *celestial poles* (P, P' in the figure), the *celestial equator* or simply *equator* ($QwQ'e$), *hour circles* (as PSE), and *parallels of declination* (as MSM').

An observer at O on the earth's surface will have his *zenith* at Z, where the plumb line at O, if produced, would meet the celestial sphere; his *horizon* is the great circle *swne*, whose pole is Z; his *meridian* is the great circle $nPZQs$, meeting the horizon in the north and south points.

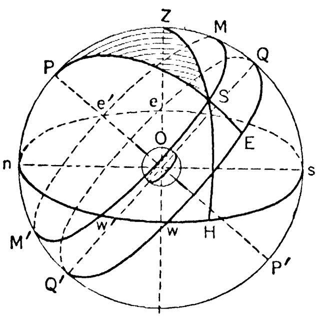


FIG. 96

Let S be a point on the celestial sphere, as the sun's center, or a star. Because of the rotation of the earth, S will appear to describe the parallel $e'MSw'M'e'$, rising at e' and setting at w'. When S has the position shown in the figure, HS is its *altitude*, denoted by h (height above horizon); $\angle\ sZH$ (measured by arc sH) is its *azimuth*, denoted by A; ZS, or $90° - h$, is the *zenith distance* of S and denoted by z. Thus h and A, or z and A, completely define the position of S with reference to horizon and zenith.

With reference to the equator and pole, ES is called the *declination* of S, denoted by δ, and $\angle\ QPE$ (angle which hour circle PS of S makes with meridian PQ) is called its *hour angle*, denoted by t; PS or $90° - \delta$ is the *polar distance* of S, and denoted by p. Thus the position of S is defined by δ and t, or by p and t.

$\triangle\ PZS$ is called the *astronomical triangle;* its parts, except the angle at S which we shall not need, are:

$$PZ = 90° - nP = 90° - \varphi;$$
$$(\varphi = \text{latitude of } O.)$$
$$PS = p = 90° - \delta;$$
$$ZS = z = 90° - h;$$
$$\angle\ ZPS = t;$$
$$\angle\ PZS = 180° - A.$$

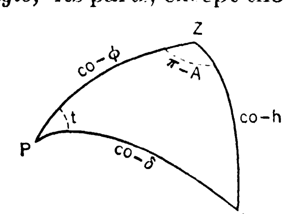


FIG. 97

141. Problems involving the astronomical triangle.

Problem 1.

To find the local mean time from an observed altitude of the sun; the latitude of the observer and the declination of the sun are assumed to be known.

To determine his local time the navigator takes a *time sight.* That is, he measures with his sextant the altitude of the sun above the horizon. This gives him side ZS in triangle PZS.

In the Nautical Almanac he can look up the declination of the sun, which gives him side PS. His known latitude gives him side PZ.

He then has three sides of the triangle from which to calculate angle ZPS, or t, the sun's hour angle. This gives him the local time.

From the local time, and the Greenwich time as shown by his chronometer at the moment when he observed the sun's altitude, he can determine his longitude.

Problem 2.

To determine the latitude by observing the altitude of the sun (or a star) when it crosses his meridian.

He starts measuring the altitude of the sun a little before local noon and continues measurements until the altitude begins to fall off. The greatest observed value is the meridian altitude.

This gives him arc sM in Fig. 96. Subtracting the sun's declination, arc QM, (or adding it if the sun is south of the equator) gives arc sQ. The complement of arc sQ is his latitude.

Problem 3.

To find the latitude by noting the time when the sun (or a star) bears due west, or due east.

This is for observation on land where the observer can point his transit due west or east and wait for the sun or star to cross the field of view.

In this case angle PZS is 90°. The time of the observation gives angle t, and the sun's declination gives side PS. Solving for side PZ gives the co-latitude.

Problem 4.

Find the hour angle and azimuth of Polaris when at greatest elongation, given the declination of the star and the latitude of the station of observation.

Consider the star's diurnal path about the pole. When the star is at *greatest elongation*, the great circle ZS (Fig. 97) is tangent to the diurnal circle, of which PS is a radius. Hence triangle PZS is right-

angled at S; PZ and PS are known, and the angles at P and Z may be found by aid of Napier's Rules.

Problem 5.

In a given latitude, and for a given declination of the sun, find the sun's hour angle at sunset and the length of day (sunrise to sunset).

Here S is on the horizon (at w' or at e') and PZS a quadrantal triangle. We obtain t by solving the polar right triangle for $180° - t$. The length of day will be $2t$.

142. EXERCISES 54

1. The meridian altitude of the sun was observed to be $61° 27'$; the sun's declination was $12° 15'$. Find the latitude.

2. The meridian altitude of Rigel was $74° 32'$; the star's declination was $-8° 16'$. Find the latitude.

3. Find the length of the longest day in latitude $60°$. The sun's declination on that day is $23° 27'$. Find the length of the shortest day in latitude $60°$. Declination is $-23° 27'$.

4. In latitude $40° 49'$ the sun's altitude is observed to be $20° 20'$; its declination is $15° 12'$; find its azimuth and hour angle.

5. With latitude and declination as in Exercise 4, find the sun's hour angle when it is due west; when it sets; find its azimuth at sunset; find the length of day.

6. With latitude and declination as in Exercise 4, find the sun's altitude and azimuth when its hour angle is $45°$.

7. The sun, in declination $12° 22'$, is observed to have an altitude of $30°$ when due west. What is the latitude of the station?

8. The declination of Polaris being $88° 49'$, find his azimuth and hour angle at greatest elongation at a station in latitude $40° 49'$.

9. As in Exercise 8 for the star 51 Cephei, $\delta = 87° 11'$, and for δ Ursæ Minoris, $\delta = 86° 37'$.

10. The stylus of a horizontal sundial consists of a rod pointing to the north celestial pole. Hence its shadow falls due north when the sun is on the meridian, that is, at apparent noon. What angle does its shadow make with the meridian one hour after apparent noon, at a place in latitude $40°$?

(*Suggestion.* In Fig. 96 let $nP = 40°$ and $\angle ZPS = 1^h$ or $15°$. The stylus lies in the line $P'P$, and its shadow, cast by the sun S, must lie in the plane $SP'P$, and hence will fall on the plane of the dial, *swne*, along the line of intersection of these two planes. This line will be determined by the center of the sphere and the point where arc SP produced will meet arc ne. Call this point S'. Then arc nS' measures the required angle, and may be found by solving right $\triangle\, nPS'$, in which $nP = 40°$ and $\angle\, nPS' = 15°$).

11. What angle does the shadow of a horizontal sundial make with its noon position t hours after noon in latitude φ?

Ans. $\tan x = \tan t \sin \varphi$, x being the required angle.

12. Calculate the angles which the hour lines of a horizontal sundial make with the noon-line in an assumed latitude.

ANSWERS TO THE ODD NUMBERED EXERCISES

Exercises 1. §4.

	sine	cosec.	cosine	secant	tangent	cotan.
1.	4/5	5/4	3/5	5/3	4/3	3/4.
3.	– 4/5	– 5/4	3/5	5/3	– 4/3	– 3/4.
5.	5/13	13/5	12/13	13/12	5/12	12/5.
7.	– 5/13	– 13/5	12/13	13/12	– 5/12	– 12/5.
9.	15/17	17/15	8/17	17/8	15/8	8/15.
11.	– 15/17	– 17/15	8/17	17/8	– 15/8	– 8/15.
13.	$3/\sqrt{13}$	$\sqrt{13}/3$	$2/\sqrt{13}$	$\sqrt{13}/2$	3/2	2/3.
15.	$-3/\sqrt{13}$	$-\sqrt{13}/3$	$2/\sqrt{13}$	$\sqrt{13}/2$	– 3/2	– 2/3.

Exercises 2. §8.

	sine	cosine	tangent	cotan.	secant	cosec.
1.	0.537	0.842	0.638	1.580	1.188	1.872.

cosines secants

3. 0.484 0.469 0.454 0.438; 2.073 2.146 2.220 2.293.

5. 45°, 45°, 45°.

Exercises 3. §11.

	α	$\sin\alpha$	$\csc\alpha$	$\cos\alpha$	$\sec\alpha$	$\tan\alpha$	$\cot\alpha$
1.	67°:	12/13	13/12	5/13	13/5	12/5	5/12.
3.	24°:	0.4	2.5	$\sqrt{21}/5$	$5/\sqrt{21}$	$2/\sqrt{21}$	$\sqrt{21}/2$.
5.	70°:	$\sqrt{8}/3$	$3\sqrt{8}/8$	1/3	3	$\sqrt{8}$	$\sqrt{8}/8$.
7.	71°:	$3\sqrt{10}/10$	$\sqrt{10}/3$	$\sqrt{10}/10$	$\sqrt{10}$	3	1/3.
9.	45°:	$\sqrt{2}/2$	$\sqrt{2}$	$\sqrt{2}/2$	$\sqrt{2}$	1	1.
11.	60°:	$\sqrt{3}/2$	$2\sqrt{3}/3$	1/2	2	$\sqrt{3}$	$\sqrt{3}/3$.
13.	73°:	$\sqrt{91}/10$	$10\sqrt{91}/91$	3/10	10/3	$\sqrt{91}/3$	$3\sqrt{91}/91$.
15.	84°:	$10\sqrt{101}/101$	$\sqrt{101}/10$	$\sqrt{101}/101$	$\sqrt{101}$	10	1/10.

Exercises 4. §14.

1. b, c, β: 142.8, 174.3, 55°. **3.** a, b, β: 55.8, 50.2, 42°. **5.** a, c, β: 470, 886, 32°. **7.** a, b, β: 0.034, 0.029, 40°. **9.** b, c, α: 21.4, 27.2, 38°. **11.** 81.9 ft. **13.** 104.6 ft. **15.** 291 ft. **17.** 35°. **19.** 10.0 in.

Exercises 6. §22.

	sine	cosine	tangent	cotangent	secant	cosecant
1.	cos 50°	– sin 50°	– cot 50°	– tan 50°	– csc 50°	sec 50°
3.	– sin 55°	– cos 55°	tan 55°	cot 55°	– sec 55°	– csc 55°
5.	– cos 85°	sin 85°	– cot 85°	– tan 85°	csc 85°	– sec 85°
7.	– sin 65°	– cos 65°	tan 65°	cot 65°	– sec 65°	– csc 65°
9.	sin 42°	cos 42°	tan 42°	cot 42°	sec 42°	csc 42°
11.	sin 50°	cos 50°	tan 50°	cot 50°	sec 50°	csc 50°
13.	sin 40°	cos 40°	tan 40°	cot 40°	sec 40°	csc 40°
15.	– cos 85°	sin 85°	– cot 85°	– tan 85°	csc 85°	– sec 85°
17.	cos 15°	sin 15°	cot 15°	tan 15°	csc 15°	sec 15°
19.	cos 20°	– sin 20°	– cot 20°	– tan 20°	– csc 20°	sec 20°
21.	sin 25°	cos 25°	tan 25°	cot 25°	sec 25°	csc 25°
23.	cos 20°	sin 20°	cot 20°	tan 20°	csc 20°	sec 20°
25.	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
27.	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2
29.	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2
31.	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$
33.	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
35.	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$
37.	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$
39.	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$

Exercises 7. §24.

	sine	cosec.	cosine	secant	tangent	cotan.
1.	$-\sqrt{2}/2$	$-\sqrt{2}$	$\sqrt{2}/2$	$\sqrt{2}$	-1	$-1.$
3.	$-\sqrt{3}/2$	$-2\sqrt{3}/3$	$1/2$	2	$-\sqrt{3}$	$-\sqrt{3}/3.$
5.	$-\sqrt{3}/2$	$-2\sqrt{3}/3$	$-1/2$	-2	$\sqrt{3}$	$\sqrt{3}/3.$
7.	$-\sqrt{2}/2$	$-\sqrt{2}$	$-\sqrt{2}/2$	$-\sqrt{2}$	1	$1.$
13.	$\sqrt{2}/2$	$\sqrt{2}$	$-\sqrt{2}/2$	$-\sqrt{2}$	-1	$-1.$

Exercises 8. §25.

	1.	**3.**	**5.**	**7.**	**9.**	**11.**
vers θ:	$(2-\sqrt{3})/2$	$1/2$	$3/2$	$(2+\sqrt{3})/2$	$3/2$	$1/2$
covers θ:	$1/2$	$(2-\sqrt{3})/2$	$(2-\sqrt{3})/2$	$1/2$	$(2+\sqrt{3})/2$	$(2-\sqrt{3})/2$
hav θ:	$(2-\sqrt{3})/4$	$1/4$	$3/4$	$(2+\sqrt{3})/4$	$3/4$	$1/4$

Exercises 9. §28.

1. 15°, 300°, $56\frac{1}{4}$°, 288°, 264°. **3.** 75°, – 85° 56′ 37″, 81° 17′ 45″, 21° 48′ 10″, 80° 48′ 35″. **5.** 16° 33′ 36″, 264° 3′, 110° 48′ 13″. **7.** $25\pi/12$, $-\pi/8$, $25\pi/24$, 1.85005, 1.63625. **9.** 0.002909, 0.000048, 0.00000048, 0.21091, 0.37703.

Exercises 10. §29.

1. $12\sqrt{3}$, 8π, $48\pi - 36\sqrt{3}$. **3.** $5\sqrt{2}$.

Exercises 11. §31.

3. Radians: 1/4, 5/4, 1/5, 1/50; degrees: 14.32, 71.62, 11.46, 1.15. **5.** 120 in., 128 in., 1700 sq. in. **7.** 0.5 rad. **9.** Angle = 0.4 rad. = 22° 55′, arc = 19.88, sector = 500, triangle = 486.7, segment = 13.3. **11.** 3 rad. **13.** 90. **15.** 270 rad./sec.

Exercises 12. §33.

1.	0.5360,	0.6350,	1.1846.	**13.**	0.84407,	− 1.5740,	− 1.8648.
3.	0.8820,	1.8715,	2.1220.	**15.**	0.88203,	1.8718,	2.1222.
5.	0.8442,	− 1.5747,	− 1.8656.	**17.**	0.43388,	0.48158,	1.1099.
7.	− 0.8820,	1.8715,	− 2.1220.	**19.**	− 0.90930,	− 2.1850,	2.4030.
9.	− 0.8442,	1.5747,	− 1.8656.	**21.**	0.97493,	4.3814,	4.4940.
11.	0.4712,	− 0.5343,	− 1.1338.	**23.**	0.54064,	0.64266,	1.1887.

Exercises 13. §35.

1. $45° + 2n\pi$, $135° + 2n\pi$. **3.** $30° + 2n\pi$, $-30° + 2n\pi$. **5.** $30° + 2n\pi$, $150° + 2n\pi$. **7.** $45° + 2n\pi$, $-45° + 2n\pi$. **9.** $45° + 2n\pi$, $-135° + 2n\pi$. **11.** $-45° + 2n\pi$, $135° + 2n\pi$. **13.** $17° 24' + 2n\pi$, $162° 36' + 2n\pi$. **15.** $31° 48' + 2n\pi$, $-148° 12' + 2n\pi$. **17.** $121° 48' + 2n\pi$, $-121° 28' + 2n\pi$.

Exercises 14. §36.

(In each case the first angle is the principal angle.)

1. 60°, 120°. **3.** 30°, − 150°. **5.** − 45°, 135°. **7.** − 63° 26′, 116° 34′. **9.** ± 75° 31′. **11.** 41° 49′, 138° 11′. **13.** ± 131° 49′. **15.** ± 126° 52′. **17.** − 41° 49′. **19.** 48° 11′. **21.** 80° 58′. **23.** − 60° 57′. **25.** 61° 38′. **27.** 138° 35′. **29.** 55° 23′.

Exercises 15. §37.

1. $30° + n\pi$, $90° + n\pi$. **3.** $51° + n\cdot 72°$, $-3° + n\cdot 72°$. **5.** $10° + n\cdot 45°$, $-20° + n\cdot 45°$. **7.** $138° 54' + 3n\pi$, $-78° 54' + 3n\pi$. **9.** $25° + n\cdot 90°$.

Exercises 16. §39.

	sin	cos	tan	csc	sec	cot
1.	$-2/3$	$\pm\sqrt{5}/3$	$\pm 2\sqrt{5}/5$	$-3/2$	$\pm 3\sqrt{5}/5$	$\pm\sqrt{5}/2$
3.	$\pm 3/5$	$\pm 4/5$	$-3/4$	$\pm 5/3$	$\pm 5/4$	$-4/3$
5.	$\pm 1/2$	$\pm\sqrt{3}/2$	$-\sqrt{3}/3$	± 2	$\pm 2\sqrt{3}/3$	$-\sqrt{3}$
7.	$\pm 40/41$	$-9/41$	$\pm 40/9$	$\pm 41/40$	$-41/9$	$\pm 9/40$
9.	$-4/5$	$\pm 3/5$	$\pm 4/3$	$-5/4$	$\pm 5/3$	$\pm 3/4$
11.	$-\frac{1}{m}$	$\pm\frac{\sqrt{m^2-1}}{m}$	$\pm\frac{1}{\sqrt{m^2-1}}$	$-m$	$\pm\frac{m}{\sqrt{m^2-1}}$	$\pm\sqrt{m^2-1}$
13.	$1+h$	$\pm\sqrt{-2h-h^2}$	$\pm\frac{1+h}{\sqrt{-2h-h^2}}$	$\frac{1}{1+h}$	$\pm\frac{1}{\sqrt{-2h-h^2}}$	$\pm\frac{\sqrt{-2h-h^2}}{1+h}$
15.	$\pm\frac{a^2-b^2}{a^2+b^2}$	$\frac{2ab}{a^2+b^2}$	$\pm\frac{a^2-b^2}{2ab}$	$\pm\frac{a^2+b^2}{a^2-b^2}$	$\frac{a^2+b^2}{2ab}$	$\pm\frac{2ab}{a^2-b^2}$

Exercises 17. §40.

3. $(1 \pm \sqrt{1-\sin^2 x})/\sin^2 x$. **5.** $2\csc^2\theta/(\csc^2\theta - 1)$.

Exercises 18. §41.

1. $n\pi$; $30° + 2n\pi$, $150° + 2n\pi$. **3.** $60° + 2n\pi$, $-60° + 2n\pi$. **5.** $45° + 2n\pi$, $225° + 2n\pi$; $-45° + 2n\pi$, $135° + 2n\pi$. (More compactly: $n\pi \pm 45°$). **7.** $n\pi$; $\pm 60° + 2n\pi$. **9.** $\pm 60° + 2n\pi$, $\pm 120° + 2n\pi$. (More compactly: $n\pi \pm 60°$.) **11.** $22°\ 30' + 2n\pi$, $202°\ 30' + 2n\pi$; $-67°\ 30' + 2n\pi$, $112°\ 30' + 2n\pi$. (More compactly: $n\pi + 22°\ 30'$; $n\pi - 67°\ 30'$.) **13.** $-126°\ 52' + 2n\pi$. **15.** $90° + 2n\pi$; $36°\ 52' + 2n\pi$. **17.** $45° + 2n\pi$, $225° + 2n\pi$; $-71°\ 34' + 2n\pi$, $108°\ 26' + 2n\pi$. **19.** $36°\ 52' + 2n\pi$.

Exercises 19. §44.

1. a, b, β: 52.02, 24.61, 25° 19′.
3. a, b, α: 2344, 1415, 58° 53′.
5. b, c, β: 2661, 3058, 60° 29′.
7. a, c, α: 1.097, 1.179, 68° 27′.
9. b, c, α: 2352, 3937, 53° 19′.
11. a, c, β: 0.0873, 0.0913, 17° 0′.
13. a, b, α: 889.0, 236.0, 75° 8′.
15. b, c, α: 0.04055, 0.05397, 41° 18′.
17. a, α, β: 52.02, 64° 41′, 25° 19′.
19. c, α, β: 3937, 53° 19, 36° 41′.
21. b, α, β: 0.0267, 73° 0′, 17° 0′,
23. c, α, β: 0.05397, 41°18′, 48° 42′.
25. b, c, β: 29.186, 37.562, 50° 59.2′.
27. a, b, α: 12758, 14247, 41° 50.7′.
29. b, c, α: 163.15, 313.04, 58° 35.3′.
31. b, α, β: 420.72, 29° 8.2′, 60° 51.8′.
33. a, c, β: 234.52, 481.67, 60° 51.8′.
35. c, α, β: 42.223, 50° 28.3′, 39° 31.7′.
37. a, b, α: 32.567, 26.873, 50° 28.3′.
39. a, α, β: 28641, 41° 31.3′, 48° 28.7′.
41. a, β, γ: 200.02, 50° 1.5′, 69° 58.5′.
43. b, α, γ: 199.77, 42° 3.7′, 81° 10.3′.
45. a, b, γ: 119.91, 209.93, 58° 50.0′.

Exercises 20. §49.

1. 17° 14′. **3.** 5° 16′. **5.** 5670 ft. **7.** 402.0 ft., 586.1 ft. **9.** 23° 26′. **13.** 809.1 in., 50360 sq. in. **15.** 144.5 ft. **17.** 34° 48′. **19.** 34.55 ft. **21.** 1418 ft.

Exercises 21. §50.

1. Proj. on OX: 100, 86.60, 70.71, 50, 0, − 86.60, − 50, 0, 50.
Proj. on OY: 0, 50, 70.71, 86.60, 100, 50, − 86.60, − 100, − 86.60.
3. Proj. on OX: 271.8, − 321.7, 271.8; on OY: 230.2, − 152.7, − 230.2.

Exercises. §52.

1. (170.5, 42° 10′). **3.** (111.2, 86° 34′). **5.** (52.9, 160° 53′). **7.** (123.0, 261° 5′).

Exercises 22. §53.

1. (144.2, N 26° 6′ E). **3.** (216.7, N 28° 2′ E). **5.** (157.1, N 33° 49′ E). **7.** (195.1, S 34° 7′ E).

Exercises 23. §54.

1. (68.4 lb., 39° 19′). **3.** (42.5 lb., 124° 15′).

Exercises 24. §56.

1. W 30° 22′ S, 111.5 miles. **5.** $BL = AB = 3$ miles. $CL = 2.30$ miles. **7.** 6.72 miles. **11.** 734 ft.

Exercises 27. §59.

1. 764 ft. **3.** 8595 ft. **7.** 3.44′. **9.** 166 in., 9980 in.. 598000 in.

Exercises 28. §61.

	1)	3)	5)	7)	9)
1.	5 $\overline{m}$.	2.5 $\overline{m}$.	6.7 $\overline{m}$.	8.6 $\overline{m}$.	33.3 $\overline{m}$.
2.	3000 yd.	750 yd.	2400 yd.	3333 yd.	2066 yd.
3.	32.	27.	135.	87.8.	81.6.

5. (a) exact, (b) exact, (c) $17\frac{7}{9}\ \overline{m}$, (d) $\dfrac{3200}{\pi}\ \overline{m}$. **7.** 40 $\overline{m}$. **9.** 31.25 $\overline{m}$.

Exercises 29. §61.

1. 610 $\overline{m}$. 133 $\overline{m}$. 533 $\overline{m}$. 1217 $\overline{m}$. 735 $\overline{m}$. **3.** 220 $\overline{m}$. 102 $\overline{m}$. 565 $\overline{m}$. 1001 $\overline{m}$. 1127 $\overline{m}$. **5.** b, c, β: 141, 188, 860 $\overline{m}$. **7.** b, c, α: 364, 1250, 1300 $\overline{m}$. **9.** a, b, α: 594, 1430, 400 $\overline{m}$. **11.** a, α, β: 640, 352 $\overline{m}$, 1048 $\overline{m}$.

Exercises 33. §70.

1. $(\sqrt{6}+\sqrt{2})/4$, $(\sqrt{6}-\sqrt{2})/4$, $2+\sqrt{3}$. **3.** 0, -1, 0. **5.** $(\sqrt{6}-\sqrt{2})/4, (\sqrt{6}+\sqrt{2})/4, 2-\sqrt{3}$. **7.** $(\sqrt{6}+\sqrt{2})/4, (\sqrt{2}-\sqrt{6})/4, -2-\sqrt{3}$. **9.** $-133/205$. **11.** $(6+4\sqrt{21})/25$, $(-6+4\sqrt{21})/25$, $(6-4\sqrt{21})/25$, $(-6-4\sqrt{21})/25$.

Exercises 35. §74.

1. $(1/2)\sqrt{2+\sqrt{3}}$, $2-\sqrt{3}$. (Compare with answers to Ex. 5, §70.) **3.** 1/2.

Exercises 36. §76.

1. $\sqrt{3}\cos 10°$. **3.** $\sin 10°$. **5.** $-2\sin 65°\sin 15°$. **7.** $\cos 10°$. **9.** $2\cos 105°\sin 35°$. **11.** $2\cos 165°\cos 115°$. **13.** $\sin 80° - \sin 40°$. **15.** $\cos 40° - \cos 80°$. **17.** $-1+\cos 100°$.

Exercises 37. §77.

1. 1, 0. **3.** 156/205, $-133/205$, $-41496/42025$, $-6647/42025$. **5.** $(\pm\sqrt{5}\pm 4\sqrt{2})/9$. **7.** $\pm\sqrt{7}(75\pm 32\sqrt{3})/111$. **9.** 204/253. **11.** 1/2.

Exercises 38. §80.

1. 32° 23′. **3.** 32° 2′. **5.** 43° 3′. **7.** 6.362. **9.** 35. **11.** 34° 3′, 44° 25; 101° 32′. **13.** 82° 49′, 55° 46′, 41° 25′. **15.** 338.3 miles.

Exercises 39. §85. (*4-place tables.*)

1. b, c, γ: 1260.6, 1069.3, 55°. **3.** a, c, α: 4.999, 7.350, 38°. **5.** a, b, α: 6758, 5802, 87° 40′. **7.** 657.8, 450.0. **9.** 1067.5, 661.1. **11.** 145.0, 110.6.

Exercises 40. §86. (*4-place tables.*)

1. α, β, c: 54° 27′, 65° 38′, 851.3. **3.** α, β, c: 26° 2′, 52° 18′, 497.5. **5.** β, γ, a: 44° 28′, 99° 24′, 3825. **7.** α, γ, b: 15° 18′, 12° 42′, 267.0. **9.** 112° 28′, 27° 32.

Exercises 41. §87.

(*5-place tables used for exercises with starred numbers.*)

1. β, γ, c: 33° 28′, 119° 14′, 59.17; β', γ', c': 146° 32′, 6° 10′, 7.285. **3.*** β, γ, b: 32° 55′, 88° 58′, 73.16; β', γ', b': 30° 52′, 91° 2′, 69.07. **5.*** α, β, a: 35° 14.7′, 21° 6.3′, 2230.9.

Exercises 42. §88. (*4-place tables.*)

1. 70° 40′, 47° 47′, 57° 33′. **3.** 104° 30′, 32° 3′, 43° 27′. **5.** No solution.

Exercises 43. §90.

(*5-place tables used for exercises with starred numbers.*)

1. a, c, γ: 3675, 5781, 70° 58′. **3.** a, b, β: 1566, 1068, 42° 27′. **5.** c, α, β: 0.1776, 76° 20′, 44° 53′. **7.** c, β, γ: 156.1, 26° 43′, 131° 56′; c', β', γ': 19.57, 153° 17′, 5° 22′. **9.** α, β, γ: 149° 49′, 3° 2′, 27° 9′. **11.*** β, γ, b: 146° 43.6′, 14° 3.7′, 3.5881. **13.** b, α, β: 0.2729, 39° 37′, 117° 51′; b', α', β': 0.0907, 140° 23′, 17° 5′. **15.** a, β, γ: 0.00251, 70° 17′, 51° 50′. **17.** b, β, γ: 0.000662, 83° 28′, 32° 42′. **19.*** a, α, β: 1.2379, 162° 18.8′, 7° 8.4′. **21.*** a, c, γ: 57285, 117600, 151° 19.6′. **23.** b, α, γ: 0.01068, 81° 51′, 55° 42′. **25.*** α, a, c: 34° 32.1′, 14261, 25100. **27.** c, β, γ: 584.1, 51° 9′, 87° 38′; c', β', γ': 100.9, 128° 51′, 9° 56′. **29.** c, β, γ: 1191, 32° 32′, 120° 10′; c', β', γ': 125.7, 147° 28′, 5° 14′. **31.*** a, β, γ: 2496.1, 100° 10.2′, 27° 38.8′. **33.*** α, γ, c: 39° 39.1′, 90° 0.0′, 18464. **35.*** β, a, b: 14° 15.5′, 0.031083, 0.010735. **37.*** γ, a, c: 32° 19.7′, 43.738, 64.587. **39.** c, α, γ: 0.005708, 79° 20′, 37° 0′; c', α', γ': 0.002561, 100° 40′, 15° 40′. **51.** 7, $\sqrt{129}$, $20\sqrt{3}$. **53.** 7/8. **55.** 45°, 60°, 75°; 612.3 ft., 683.0 ft. **57.** 261.4. **71.** 1.239 mi. **73.** 1066 ft. **75.** $40\sqrt{5}$ ft. **77.** 45° 3′. **79.** 698.3 ft. **81.** 22.3, 70.6 ft. **85.** 62 ft. **87.** 1142 ft. **91.** 25, $33\frac{1}{3}$, $41\frac{2}{3}$ ft. **93.** 37.5 ft. **95.** 28° 57′, 46° 34′, 104° 29′; 5.892, 8.838, 11.784. (The exact values of the sides are $20\sqrt{2/23}$, $30\sqrt{2/23}$, $40\sqrt{2/23}$.) **97.** 27.35 ch.; 97.46 A. **99.** 14.4 ch. north of AB. **101.** 718.7 lb. **103.** 2.51 sec. **105.** 48° 53′. **107.** *Total defl.* $= (i - r) + (i' - r')$, where $r = \text{Sin}^{-1}\left(\frac{\sin i}{\mu}\right)$, $r' = \alpha - r$, and $i' = \text{Sin}^{-1}(\mu \sin r')$.

Exercises 44. §94.

1. $30°$, $\pi/6$. **3.** $-90°$, $-\pi/2$. **5.** $60°$, $\pi/3$. **7.** $-30°$, $-\pi/6$. **9.** $30°$, $\pi/6$. **11.** $90°$, $\pi/2$. **13.** $150°$, $5\pi/6$. **15.** $90°$, $\pi/2$. **17.** $30°$, $\pi/6$. **19.** $-60°$, $-\pi/3$. **21.** $78° 27'$. **23.** $54° 44'$. **25.** $126° 52'$. **27.** $71° 34'$. **29.** $67° 30'$. **31.** $-33° 41'$. **33.** $53° 8'$. **35.** $-76° 43'$. **37.** $-18° 26'$. **39.** $53° 8'$. **41.** $3/\sqrt{10}$. **43.** $\sqrt{0.9}$. **45.** $5/3$. **47.** $0.4\sqrt{5}$. **49.** $\sqrt{10}/10$. **51.** $-4\sqrt{5}$. **53.** $\sqrt{3}/3$. **55.** $3/2$. **57.** $-8/17$. **59.** $0.4\sqrt{5}$.

Exercises 46. §106.

1. $\sqrt{2}$, $-45°$; 5, Arctan $(3/4)$; $\sqrt{146}$, $\pi +$ Arctan $(-11/5)$; 2, $90°$; 2, $0°$; 2, $0°$; 6, $30°$; 36, $-60°$; 4, $90°$.

Exercises 47. §108.

3. ± 3, $\pm 3i$. **5.** $x_1 = 2$; $x_2 = 2(\cos 72° + i \sin 72°)$; $x_3 = 2(\cos 144° + i \sin 144°)$; etc. **7.** $x = \sqrt{3}\,(\cos n\,60° + i \sin n\,60°)$, $n = 0, 1, 2, 3, 4, 5$, or, x_1, x_2, x_3, etc., $= \sqrt{3}$, $(\sqrt{3} + 3i)/2$, $(-\sqrt{3} + 3i)/2$, etc.

Exercises 49. §119.

1. c, a, β: $112° 44'$, $133° 28'$, $67° 50'$. **3.** a, b, c: $4° 3'$, $44° 19'$, $44° 29'$. **5.** b, α, β: $40° 39'$, $122° 38'$, $50° 16'$. **7.** a, b, α: $146° 34'$, $109° 48'$, $144° 57'$. **9.** No solution. **11.*** a, b, β: $32° 3.4'$, $138° 17.0'$, $120° 46.1'$. **13.** α, β, γ: $129° 59'$, $36° 54'$, $59° 3'$. **15.** b, α, γ: $78° 11'$, $13° 51'$, $129° 42'$. **17.** b, α, β: $84° 54'$, $108° 28'$, $84° 37'$. **19.*** a, b, α: $28° 46.5'$, $63° 57.3'$, $12° 41.7'$. **21.*** a, α, γ: $122° 17.5'$, $132° 15.8'$, $118° 53.9'$.

Exercises 50. §131.

1. α, β, γ: $81° 39'$, $70° 10'$, $64° 47'$. **3.** α, β, γ: $140° 0'$, $61° 40'$, $26° 30'$. **5.** b, α, γ: $117° 5'$, $65° 30'$, $123° 21'$. **7.** a, β, γ: $82° 7.0'$, $111° 32.8'$, $92° 28.4'$. **9.** b, c, α: $157° 40'$, $33° 20'$, $62° 51'$. **11.** b, c, α: $134° 55.3'$, $62° 47.7'$, $111° 39.6'$. **13.** a, b, c: $163° 34'$, $169° 40'$, $8° 11.6'$. **15.** a, b, c: $49° 24'$, $149° 34.4'$, $148° 33.5'$. **17.** a, α, γ: $118° 20'$, $136° 57'$, $29° 40'$. **19.** c, α, γ: $153° 38.7'$, $42° 37.3'$, $160° 1.4'$; or c', α', γ': $90° 5.7'$, $137° 22.7'$, $50° 18.9'$. **21.** b, c, β: $124° 59.4'$, $33° 22'$, $83° 25.6'$. **23.** b, c, γ: $57° 35'$, $154° 15.5'$, $151° 15'$; or b', c', γ': $122° 25'$, $64° 2.2'$, $84° 41.7'$. **25.** c, α, β: $48° 46.4'$, $121° 28.6'$, $28° 33.3'$. **27.** α, β, γ: $53° 38.8'$, $118° 15.8'$; $112° 14.2'$. **29.** b, α, γ: $125° 30.9'$, $34° 59.3'$, $33° 11.6'$. **31.** a, b, γ: $23° 57.2'$, $118° 2.2'$, $102° 5.8'$. **33.** a, b, c: $165° 25.3'$, $14° 34.7'$, $168° 47.2'$. **35.** a, α, β: $18° 48.7'$, $53° 38.8'$, $118° 15.8'$. **37.** b, α, β: $118° 2.2'$. $27° 22.6'$, $91° 26.7'$. **39.** a, α, γ: $152° 43.8'$, $153° 17.1'$, $78° 15.8'$.

Exercises 51. §133.

1. Latitudes: $43° 01'$, $31° 47'$, $25° 32'$; longitudes: $132° 12'$, $147° 46'$, $154° 04'$; courses: S $52° 43'$ W, S $43° 11'$ W, S $40° 08'$ W. **3.** $137° 11'$ W; 788 naut. miles. **5.** 6829 naut. miles. **7.** $168° 38'$ W; S $42° 59'$ W; 3410 naut. miles.

Exercises 52. §136.

1. N. Y.–S. F. 2568 statute miles, N. Y.–M. C. 2090 s.m., S. F.–M. C. 1889 s.m.; angles: N. Y. 48° 58′, S. F. 55° 48′, M. C. 82° 40′; area 2025300 sq. miles. **3.*** 7929.1 naut. mi. **5.*** 23° 36.3′ N, 145° 6.7′ W; S 48° 31.3′ W. **7.*** 1617.3 naut. mi., C at Manila S 35° 13.2′ W, C at Tokyo S 43° 22.8′ W. **9.** 1380 n.m., 1436 n.m.

Exercises 54. §142.

1. 40° 48′. **3.*** 18 h. 33 m. 50 s., 10 h. 0 m. 21 s. **5.*** 18 h. 33 m. 50 s., 110° 16.1′, 13 h. 48 m. 34 s. **7.** 25° 21′. **9.** At western elongation: 176° 17′, 5 h. 50 m. 16 s.; 175° 32′, 5 h. 48 m. 16 s.

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APPENDIX

A

The Greek Alphabet

Letters	Name	Letters	Name	Letters	Name
A, α,	Alpha	I, ι,	Iota	P, ρ,	Rho
B, β,	Beta	K, κ,	Kappa	Σ, σ,	Sigma
Γ, γ,	Gamma	Λ, λ,	Lambda	T, τ,	Tau
Δ, δ,	Delta	M, μ,	Mu	Υ, υ,	Upsilon
E, ϵ,	Epsilon	N, ν,	Nu	Φ, ϕ,	Phi
Z, ζ,	Zeta	Ξ, ξ,	Xi	X, χ,	Chi
H, η,	Eta	O, o,	Omicron	Ψ, ψ,	Psi
Θ, θ,	Theta	Π, π,	Pi	Ω, ω,	Omega

Formulas of Plane Trigonometry

Definitions. — In right triangle ABC, whose sides are a, b, c

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b},$$

$$\csc A = \frac{c}{a}, \quad \sec A = \frac{c}{b}, \quad \cot A = \frac{b}{a}.$$

More generally, if x be an angle of any magnitude, as XOP in figure 4,

$$\sin x = \frac{\text{ordinate}}{\text{distance}}, \quad \cos x = \frac{\text{abscissa}}{\text{distance}}, \quad \tan x = \frac{\text{ordinate}}{\text{abscissa}},$$

$$\csc x = \frac{\text{distance}}{\text{ordinate}}, \quad \sec x = \frac{\text{distance}}{\text{abscissa}}, \quad \cot x = \frac{\text{abscissa}}{\text{ordinate}}.$$

$$\text{vers } x = 1 - \cos x. \quad \text{covers } x = 1 - \sin x.$$

$$\text{haversine of } x = \text{hav } x = \tfrac{1}{2} \text{ vers } x = \frac{1 - \cos x}{2}.$$

Relations between the functions of an angle. Formulas, Group A. §19.

1. $\sin x = \dfrac{1}{\csc x}.$ 3. $\tan x = \dfrac{1}{\cot x}.$ 5. $\cot x = \dfrac{\cos x}{\sin x}.$

2. $\cos x = \dfrac{1}{\sec x}.$ 4. $\tan x = \dfrac{\sin x}{\cos x}.$ 6. $\sin^2 x + \cos^2 x = 1.$

7. $1 + \tan^2 x = \sec^2 x.$ 8. $1 + \cot^2 x = \csc^2 x.$

Rules for expressing any function of any angle in terms of a function of an acute angle. §21.

Any function of any angle x is numerically equal to the $\begin{cases}\text{same function}\\ \text{co-function}\end{cases}$ of x increased or diminished by any $\begin{cases}\text{even}\\ \text{odd}\end{cases}$ multiple of 90°.

The sign of the result must be determined according to the quadrant of x.

Functions of $+x$ and $-x$. §23.

$f(+x) = f(-x)$, when f = cosine or secant.

$f(+x) = -f(-x)$, when f = sine, cosecant, tangent, cotangent.

Angles corresponding to a given function. §34.

Let θ_1 and θ_2 be the basic angles corresponding to a given value of a function. Then all angles are $\theta_1 + 2n\pi$ and $\theta_2 + 2n\pi$, where n is any integer, positive or negative, or zero. In exceptional cases there may be only one basic angle.

Formulas, Group B. §69.

9. $\sin(x + y) = \sin x \cos y + \cos x \sin y.$

10. $\cos(x + y) = \cos x \cos y - \sin x \sin y.$

11. $\sin(x - y) = \sin x \cos y - \cos x \sin y.$

12. $\cos(x - y) = \cos x \cos y + \sin x \sin y.$

13. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$

14. $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}.$

15. $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

16. $$\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

Formulas, Group C. §73.

Double Angle.

17. $\sin 2x = 2 \sin x \cos x.$

18. $\cos 2x = \cos^2 x - \sin^2 x,$

$= 1 - 2 \sin^2 x,$

$= 2 \cos^2 x - 1.$

19. $$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

Half-Angle.

20. $$\sin \tfrac{1}{2}x = \pm \sqrt{\frac{1 - \cos x}{2}}.$$

21. $$\cos \tfrac{1}{2}x = \pm \sqrt{\frac{1 + \cos x}{2}}.$$

22. $$\tan \tfrac{1}{2}x = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}, \quad = \frac{1 - \cos x}{\sin x}, \quad = \frac{\sin x}{1 + \cos x}.$$

Formulas, Group D. §75.

23. $$\sin u + \sin v = 2 \sin \frac{u + v}{2} \cos \frac{u - v}{2}.$$

24. $$\sin u - \sin v = 2 \cos \frac{u + v}{2} \sin \frac{u - v}{2}.$$

25. $$\cos u + \cos v = 2 \cos \frac{u + v}{2} \cos \frac{u - v}{2}.$$

26. $$\cos u - \cos v = -2 \sin \frac{u + v}{2} \sin \frac{u - v}{2}.$$

Solution of right triangles. Solve by means of the definitions of the trigonometric functions.

Oblique plane triangles. Formulas, Group E.

1. *Law of Sines:* $a : b : c = \sin \alpha : \sin \beta : \sin \gamma$ §78.
2. *Law of Cosines:* $a^2 = b^2 + c^2 - 2bc \cos \alpha.$ §79.
3. *Law of Tangents:* $\dfrac{a - b}{a + b} = \dfrac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}.$ §81.

Half-angles. §82.

Let $s = \frac{1}{2}(a + b + c)$ and $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

4. $\sin \frac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}}$. **6.** $\tan \frac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

5. $\cos \frac{1}{2}\alpha = \sqrt{\frac{s(s-a)}{bc}}$. **7.** $\tan \frac{1}{2}\alpha = \frac{r}{s-a}$.

Area. §89.

8. $K = \frac{1}{2}ab \sin \gamma$. **9.** $K = rs$.

Solution of oblique plane triangles.

Case I. Given two angles and a side. §85.
Use law of sines.

Case II. Given two sides and the included angle. §86.
Use law of tangents, then law of sines

Case III. Given two sides and an opposite angle. §87.
Use law of sines. Ambiguous case.

Case IV. Given the three sides. §88.
Use one of the formulas (4), (5), (6), or (7).

Formulas of Spherical Trigonometry

Spherical right triangle. §114–§118. — Let α, β, γ, be the angles and a, b, c the sides. Arrange the five parts a, b, co-β, co-c, co-α in circular order. These parts are then connected by Napier's Rules:

$$\text{sine of middle part} = \begin{cases} \text{product of cosines of opposite parts;} \\ \text{product of tangents of adjacent parts.} \end{cases}$$

To solve a spherical right triangle use Napier's Rules to write a formula involving the two given parts and a required part.

To solve a quadrantal triangle, solve its polar right triangle.

Spherical oblique triangles. Formulas, Group F.

1. *Law of Sines:* $\sin a : \sin b : \sin c = \sin \alpha : \sin \beta : \sin \gamma$.
§120.

2. *Law of Cosines:* $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$ §120.

3. $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a.$ §121.

Half-angles. §122.

$$s = \tfrac{1}{2}(a + b + c);\ \tan r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}.$$

5. $$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$

6. $$\cos \tfrac{1}{2}\alpha = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$$

7. $$\tan \tfrac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}.$$

9. $$\tan \tfrac{1}{2}\alpha = \frac{\tan r}{\sin (s - a)}.$$

Half-sides. §123.

$$S = \tfrac{1}{2}(\alpha + \beta + \gamma);\ \tan R = \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}}.$$

14. $$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S - \alpha)}{\sin \beta \sin \gamma}}.$$

15. $$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos (S - \beta) \cos (S - \gamma)}{\sin \beta \sin \gamma}}.$$

16. $$\tan \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S - \alpha).}{\cos (S - \beta) \cos (S - \gamma)}}.$$

17. $$\tan \tfrac{1}{2}a = \tan R \cos (S - \alpha).$$

Napier's analogies. §124.

20′. $$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)} \tan \tfrac{1}{2}c.$$

21′. $$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)} \tan \tfrac{1}{2}c.$$

22′. $$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}\gamma.$$

23′. $$\tan \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}\gamma.$$

Delambre's or Gauss's analogies. §125.

24. $$\sin \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}c} \cos \tfrac{1}{2}\gamma.$$

25. $$\sin \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}c} \cos \tfrac{1}{2}\gamma.$$

26. $$\cos \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}c} \sin \tfrac{1}{2}\gamma.$$

27. $$\cos \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}c} \sin \tfrac{1}{2}\gamma.$$

Case V. Alternative method. Given, *b, c,* α; calculate *a,* β, γ. §128.

28. $\tan m = \cos \alpha \tan b;\ \tan n = \cos \alpha \tan c.$

29. $\cos a = \cos b \sec m \cos (c - m) = \cos c \sec n \cos (b - n).$

30. $\cot \beta = \cot \alpha \csc m \sin (c - m).$

31. $\cot \gamma = \cot \alpha \csc n \sin (b - n).$

Haversine formulas. §129.

32. $$\operatorname{hav} \alpha = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}.$$

33. $$\operatorname{hav} \alpha = \frac{\operatorname{hav} a - \operatorname{hav} (b - c)}{\sin b \sin c}.$$

34. $$\operatorname{hav} a = \operatorname{hav} (b - c) + \sin b \sin c \operatorname{hav} \alpha.$$

Spherical excess. Area. §126.

$$E = (\alpha + \beta + \gamma) - 180°.$$

$$\tan \tfrac{1}{2}E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin \gamma}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos \gamma}.$$

$$\tan \tfrac{1}{4}E = \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)}.$$

$$\text{Area} = \frac{E \text{ (degrees)}}{720} \times 4\pi R^2 = E \text{ (radians)} \times R^2.$$

Solution of spherical oblique triangle. §§127–9.

I. Given two sides and an opposite angle.

Use law of sines, then Napier's Analogies. Two solutions possible.

II. Given two angles and an opposite side.
As in I.

III. Given the three sides.
Use formulas for the half-angles.

IV. Given the three angles.
Use formulas for the half-sides.

V. Given two sides and their included angle.
Use Napier's Analogies, 22′ and 23′, then law of sines.

VI. Given two angles and their included side.
Use Napier's Analogies, 20′ and 21′, then law of sines.

Alternative method under Case V. §128.

Haversine method. §129.

Vertex method. §134.

APPENDIX

B

EXPLANATION OF THE TABLES AND THEIR USE

TABLE I

Common logarithms. Definition. *The common logarithm of a number is the exponent which must be applied to 10 to produce the given number.*

The symbol for the common logarithm of a number n is $\log_{10} n$, which is read:

" *The logarithm of n to the base 10.*"

Examples.

$$\left.\begin{array}{l} 10^2 = 100 \\ 10^3 = 1000 \\ 10^0 = 1 \\ 10^{-1} = 0.1 \\ 10^{-2} = 0.01 \end{array}\right. \quad \therefore \text{ common logarithm of } \left\{\begin{array}{llll} 100 = 2, & \text{or} & \log_{10} 100 = 2. \\ 1000 = 3, & \text{or} & \log_{10} 1000 = 3. \\ 1 = 0, & \text{or} & \log_{10} 1 = 0. \\ 0.1 = -1, & \text{or} & \log_{10} 0.1 = -1. \\ 0.01 = -2, & \text{or} & \log_{10} 0.01 = -2. \end{array}\right.$$

In these equations 10 is called the **base** of the system of logarithms. Other numbers might be used as bases, but for purposes of computation the base in common use is 10.

In general, if $$n = 10^x,$$

then *the common logarithm of* $n = x$, or $\log_{10} n = x$.

Theory of logarithms. So much of the theory of logarithms as is required in ordinary computation may be summed up in the following rules:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

$$\log_{10} m \cdot n = \log_{10} m + \log_{10} n.$$

II. *The logarithm of a fraction equals the logarithm of the numerator minus the logarithm of the denominator.*

$$\log_{10} \frac{m}{n} = \log_{10} m - \log_{10} n.$$

III. *The logarithm of the pth power of a number equals p times the logarithm of the number.*

$$\log_{10} m^p = p \log_{10} m.$$

Proofs.

I. Given two numbers m and n whose common logarithms are x and y respectively.

That is $\log_{10} m = x$ and $\log_{10} n = y.$

Then by definition of logarithms,

$$m = 10^x \quad \text{and} \quad n = 10^y.$$

Hence $m \cdot n = 10^x \cdot 10^y = 10^{x+y}.$

Therefore $\log_{10} m \cdot n = x + y = \log_{10} m + \log_{10} n.$

II. Proceeding as in I except that we divide m by n, we have

$$\frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y}.$$

Therefore $\log_{10} \frac{m}{n} = x - y = \log_{10} m - \log_{10} n.$

III. To prove that $\log_{10} m^p = p \log_{10} m$, let x be the common logarithm of m.

That is $\log_{10} m = x.$

Then $m = 10^x.$

Raising to pth power: $m^p = (10^x)^p = 10^{px}.$

Therefore, by definition of a logarithm,

$$\log_{10} m^p = px = p \log_{10} m.$$

This proof holds whether p is an integer or not. In applying the formula roots are to be written as fractional exponents, thus:

$$\log_{10} \sqrt[3]{m^2} = \log_{10} m^{\frac{2}{3}} = \tfrac{2}{3} \log_{10} m.$$

Exercises. Prove:

1. $\log_{10} mnr = \log_{10} m + \log_{10} n + \log_{10} r.$

2. $\log_{10} \dfrac{mn}{rs} = \log_{10} m + \log_{10} n - \log_{10} r - \log_{10} s.$

3. $\log_{10} m^p n^q = p \log_{10} m + q \log_{10} n.$

4. $\log_{10} \dfrac{m^p}{n^q} = p \log_{10} m - q \log_{10} n.$

5. $\log_{10} \sqrt{m^3 n^5} = \tfrac{3}{2} \log_{10} m + \tfrac{5}{2} \log_{10} n.$

6. $\log_{10} \sqrt[3]{\dfrac{m^2 n}{r^4}} = \tfrac{2}{3} \log_{10} m + \tfrac{1}{3} \log_{10} n - \tfrac{4}{3} \log_{10} r.$

7. $\log_{10} \dfrac{(mn)^3}{\sqrt{r^5 s^3}} = 3 \log_{10} m + 3 \log_{10} n - \tfrac{5}{2} \log_{10} r - \tfrac{3}{2} \log_{10} s.$

The proofs of rules I, II, III are also valid when the base 10 is replaced by any other positive number. In what follows we deal exclusively with the base 10, and hence we shall usually omit the subscript 10, so that *$log_{10}\, m$* will be written merely *log m*.

Numerous applications of these rules will be found in the explanation of the use of Table I.

Table of common logarithms. If we ask the question — What power of 10 will give 302? — we can see at once that the answer must lie between 2 and 3, because 302 lies between 10^2 and 10^3. That is, $302 = 10^{2.+}$, and $\log_{10} 302 = 2.+$.

The necessary decimal can be supplied by reference to a *table of logarithms*, such as Table I.

The function of such a table is to furnish the *decimal part* of the common logarithm of any number. The tables in this text give these decimals to four places. For more accurate computations 5-place, 6-place, and 7-place tables are in common use. The integral part of the logarithm is to be supplied by the computer.

Definitions. *The integral part of a logarithm is called the* **characteristic,** *and the decimal part the* **mantissa.**

Rules for characteristics.

(**a**) When the number has n significant figures to the left of the decimal point, the characteristic of its logarithm is $n - 1$.

(**b**) When the number is a decimal with n ciphers between the decimal point and the first digit which is not zero, the characteristic of its logarithm is $9 - n$, and -10 must be supplied to complete the logarithm.

The reason for these rules will become evident when we consider an example.

Example. Let us find log 302. In the table find 30 in the left-hand column and run across the page horizontally to the column headed 2. There we find that mantissa of log 302 = .4800. Now 302 lies between 100 and 1000, i.e. between 10^2 and 10^3. Hence, by the definition of a logarithm, log 302 must lie between 2 and 3. Therefore the characteristic is 2, and

$$\log 302 = 2.4800.$$

This is of course not the *exact* logarithm of 302, but only its value to four decimal places.

Writing the last equation in exponential form, we have

$$302 = 10^{2.4800}.$$

Multiplying both sides by 10,

$3020 = 10 \times 10^{2.4800} = 10^{3.4800}$. Hence, $\log 3020 = 3.4800$.

Multiplying again by 10,

$30200 = 10 \times 10^{3.4800} = 10^{4.4800}$. Hence, $\log 30200 = 4.4800$.

Therefore, where a number is multiplied by 10, the characteristic of its logarithm is increased by 1; the mantissa remains unchanged.

Dividing the above equation successively by 10, we obtain

$$\begin{aligned}
30.2 &= 10^{2.4800} \div 10 = 10^{1.4800},\\
3.02 &= 10^{1.4800} \div 10 = 10^{0.4800},\\
.302 &= 10^{0.4800} \div 10 = 10^{0.4800-1},\\
.0302 &= 10^{0.4800-1} \div 10 = 10^{0.4800-2},\\
.00302 &= 10^{0.4800-2} \div 10 = 10^{0.4800-3},
\end{aligned}$$

and so on. As logarithmic equations these are:

$$\begin{aligned}
\log 30.2 &= 1.4800,\\
\log 3.02 &= 0.4800,\\
\log .302 &= 0.4800 - 1 = 9.4800 - 10,\\
\log .0302 &= 0.4800 - 2 = 8.4800 - 10,\\
\log .00302 &= 0.4800 - 3 = 7.4800 - 10,
\end{aligned}$$

and so on. The second form in the last three equations is used for convenience in computations; it is in accordance with rule (**b**).

To discuss rules (**a**) and (**b**) more generally, let m be any number. Then by the definition of a logarithm, when

	m lies between	$\log m$ lies between
(1)	1 and 10,	0 and 1,
(2)	10 and 100,	1 and 2,
(3)	100 and 1000,	2 and 3,
(4)	1000 and 10000,	3 and 4,

and so on. Therefore, when m has

(1) 1 digit to the left of the point, $\log m = 0.+ \cdots;$
(2) 2 digits to the left of the point, $\log m = 1.+ \cdots;$
(3) 3 digits to the left of the point, $\log m = 2.+ \cdots;$
(4) 4 digits to the left of the point, $\log m = 3.+ \cdots;$

and so on. Hence rule (**a**).

In the case of decimal numbers,

	when m lies between	$\log m$ lies between
(1)	1 and 0.1,	0 and -1,
(2)	0.1 and 0.01,	-1 and -2,
(3)	0.01 and 0.001,	-2 and -3,
(4)	0.001 and 0.0001,	-3 and -4,

and so on. That is, when m is a decimal number in which

(1) no cipher follows the point, $\log m = 9.+ \cdots - 10;$
(2) 1 cipher follows the point, $\log m = 8.+ \cdots - 10;$
(3) 2 ciphers follow the point, $\log m = 7.+ \cdots - 10;$
(4) 3 ciphers follow the point, $\log m = 6.+ \cdots - 10;$

and so on. Hence rule (**b**).

Interpolation. — *Example.* Find log 3024.

From the table,

$$\left.\begin{array}{l}\text{mantissa of } \log 302 = .4800; \\ \text{mantissa of } \log 303 = .4814;\end{array}\right\} \text{difference} = .0014.$$

Assuming that the increase in the logarithm is proportional to the increase in the number, we have

$$\text{mantissa of } \log 3024 = .4800 + .4 \times .0014 = .4806.$$

The result is here given to the nearest unit in the fourth decimal place, $.4 \times .0014$ being taken equal to .0006 in place of .00056.

Proportional parts. For convenience in interpolation, the tabular differences greater than 20 are subdivided into tenths and tabulated under the heading " Prop. Parts." When the difference is less than 20, the interpolation is best made mentally. If it is desired, the table of proportional parts may be used when $d < 20$ by taking half the proportional part corresponding to double the difference.

Examples.

1. $\log 164.3 = ?$

$$\begin{array}{rl}\text{Mantissa of } \log 164 = & .2148; \ d = 27, \\ \text{Correction for } .3 = & \underline{8} \\ \log 164.3 = & 2.2156\end{array}$$

2. $\log (164.3)^{\frac{2}{3}} = ?$

$$\begin{aligned}\log (164.3)^{\frac{2}{3}} &= \tfrac{2}{3} \log 164.3, \\ &= \tfrac{2}{3}(2.2156) = 1.4771.\end{aligned}$$

3. $\log .01047 = ?$

$$\begin{array}{rl}\text{Mantissa of } \log 104 = & .0170; \quad d = 42, \\ \text{Correction for } .7 = & \underline{29} \\ \log .01047 = & 8.0199 - 10\end{array}$$

4. $\log \sqrt[3]{(.01047)^4} = ?$

$$\begin{aligned}\sqrt[3]{.01047^4} &= (.01047)^{\frac{4}{3}}, \\ \log \sqrt[3]{(.01047)^4} &= \tfrac{4}{3} \log (0.01047), \\ &= \tfrac{4}{3}\,(8.0199 - 10).\end{aligned}$$

$$4(8.0199 - 10) = 32.0796 - 40 = 22.0796 - 30.$$

$$\tfrac{1}{3}(22.0796 - 30) = 7.3599 - 10.$$

Note. When a logarithm which is followed by -10 is to be divided by a

number, add and subtract a multiple of ten so that the quotient will come out in a form followed by -10. Thus:

$$\tfrac{1}{4}(8.2448 - 10) = \tfrac{1}{4}(38.2448 - 40) = 9.5612 - 10.$$

Anti-logarithm. The number whose logarithm is x is called the *anti-logarithm* of x.

Thus, if $x = \log m$, then $m =$ anti-log x.

Given a logarithm, to obtain the corresponding number (anti-logarithm).

Examples.

1. $$\log m = 0.4806. \quad m = ?$$

The given logarithm lies between the tabular logarithms .4800 and .4814, to which correspond the numbers 302 and 303 respectively. Thus we have

$$\begin{array}{cl} \text{Number.} & \text{Mantissa of log.} \\ 302 & \left.\begin{array}{l} \left.\begin{array}{l} .4800 \\ .4806 \end{array}\right\} 6 \\ .4814 \end{array}\right\} 14 \\ m & \\ 303 & \end{array}$$

Hence, without regard to the decimal point, $m = 302 + \frac{6}{14} = 3024+$. Pointing off properly,

$$m = \text{anti-log } 0.4806 = 3.024+.$$

2. $$\log m = 7.0959 - 10. \quad m = ?$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \text{mantissa of log } 124 = .0934 \\ \text{mantissa of log } \ \ m = .0959 \end{array}\right\} 25 \\ \text{mantissa of log } 125 = .0969 \end{array}\right\} 35$$

Hence m has the sequence of figures

$$124 + \tfrac{25}{35} = 1247+.$$

Pointing off properly,

$$m = \text{anti-log } (7.0959 - 10) = .001247+.$$

Note. The value of the quotient $\frac{25}{35}$ may be obtained from the column of Prop. Parts by finding the number of tenths of 35 required to equal 25. We have from this column,

$$.7 \times 35 = 24.5 \text{ and } .8 \times 35 = 28.0.$$

Hence we see that to make 25 we need a little more than $.7 \times 35$. A close approximation would be $.71+$, making $m = .0012471+$.

When the tabular difference is large, it is possible to obtain correctly more than four significant figures of a number when its four-place logarithm is given.

Cologarithm. The *cologarithm* of a number is the logarithm of the reciprocal of the number.

Thus: $\operatorname{colog} m = \log \frac{1}{m} = \log 1 - \log m = -\log m.$

In practice we usually write it in the form

$$\operatorname{colog} m = -\log m = (10 - \log m) - 10.$$

Rule. To form the cologarithm of a number, subtract its logarithm from 10 and write -10 after the result.

Examples.

1.
$$\begin{aligned}\operatorname{colog} 302 &= (10 - \log 302) - 10\\ &= (10 - 2.4800) - 10 = 7.5200 - 10.\end{aligned}$$

2.
$$\begin{aligned}\operatorname{colog} .003024 &= (10 - \log .003024) - 10\\ &= (10 - [7.4806 - 10]) - 10 = 2.5194.\end{aligned}$$

Use of the cologarithm.

Example. Calculate the value of $\dfrac{302 \times .415}{541 \times .0828}$.

Let m be the value of the given fraction. Then without the use of cologarithms the calculation is as follows.

$$\log m = \log 302 + \log .415 - \log 541 - \log .0828.$$

$$\begin{array}{rlrl} \log 302 = & 2.4800 & \log 541 = & 2.7332 \\ \log .415 = & \underline{9.6180 - 10} & \log .0828 = & \underline{8.9180 - 10} \\ & 12.0980 - 10 & & 11.6512 - 10 \\ & \underline{11.6512 - 10} & & \\ \log m = & 0.4468, & m = & 2.7975. \end{array}$$

To use cologarithms, we write

$$m = 302 \times .415 \times \frac{1}{541} \times \frac{1}{.0828}.$$

$$\begin{aligned} \log m = \log 302 + \log .415 &+ \operatorname{colog} 541 + \operatorname{colog} .0828\\ \log 302 &= 2.4800\\ \log .415 &= 9.6180 - 10\\ \operatorname{colog} 541 &= 7.2668 - 10\\ \operatorname{colog} .0828 &= \underline{1.0820 \qquad\quad}\\ \log m &= 20.4468 - 20\\ m &= 2.7975. \end{aligned}$$

As a last example, we calculate the value of the quantity,

$$m = \sqrt{\frac{(.00812)^{\frac{2}{3}} \times (-471.2)^3}{(-522.3)^3 \times (.01242)^{\frac{3}{4}}}}.$$

To take account of the signs, which must be done independently of the logarithmic calculation, we note that the cube of a

negative quantity occurs on both sides of the fraction; hence the sign of the fraction is plus.

We now write

$$\log m = \tfrac{1}{2}[\log (.00812)^{\frac{2}{3}} + \log (471.2)^3 + \text{colog } (522.3)^3 + \text{colog } (.01242)^{\frac{3}{4}}].$$

$$\begin{array}{ll} \log .00812 = 7.9096 - 10 & \log (.00812)^{\frac{2}{3}} = 8.6064 - 10 \\ \log 471.2 = 2.6732 & \log (471.2)^3 = 8.0196 \\ \log 522.3 = 2.7179 & \log (522.3)^3 = 8.1537 \\ \log .01242 = 8.0941 - 10 & \log (.01242)^{\frac{3}{4}} = 8.5706 - 10 \end{array}$$

Hence

$$\begin{array}{rl} \log (.00812)^{\frac{2}{3}} = & 8.6064 - 10 \\ \log (471.2)^3 = & 8.0196 \\ \text{colog } (522.3)^3 = & 1.8463 - 10 \\ \text{colog } (.01242)^{\frac{3}{4}} = & 1.4294 \\ 2 | & 19.9017 - 20 \\ \log m = & 9.9508 - 10 \\ m = & .8929. \end{array}$$

Exercises. Verify the following equations:

1. $\log 7 = 0.8451$.
2. $\log 253 = 2.4031$.
3. $\log 253.5 = 2.4040$.
4. $\log .0253 = 8.4031 - 10$.
5. $\log .002533 = 7.4036 - 10$.
6. $\log 6544 = 3.8158$.
7. $\log 4.007 = 0.6028$.
8. $\log .9995 = 9.9998 - 10$.
9. $\log \sqrt{766} = 1.4421$.
10. $\log \frac{1}{766} = 7.1158 - 10$.
11. $\log (.0022)^3 = 2.0272 - 10$.
12. $\log \sqrt[3]{.0022} = 9.1141 - 10$.
13. $\log (.01401)^{\frac{4}{5}} = 8.5171 - 10$.
14. $\log (.0003684)^{\frac{7}{2}} = 7.9820 - 20$.
15. $\text{colog } 200 = 7.6990 - 10$.
16. $\text{colog } .7 = 0.1549$.
17. $\text{colog } .0448 = 1.3487$.
18. $\text{colog } \sqrt{5475} = 8.1308 - 10$.
19. $\text{colog } (.0003684)^{\frac{7}{2}} = 12.0180$.
20. $\text{antilog } 1.2222 = 16.68$.
21. $\text{antilog } 3.6675 = 4650$.
22. $\text{antilog } 0.4000 = 2.5118$.
23. $\text{antilog } (8.3250 - 10) = .021135$.
24. $\text{antilog } (6.9525 - 10) = .0008964$.
25. $(.748)^3 = .4185$.
26. $\sqrt[3]{-.0822} = -.4348$.
27. $(-6.213)^{\frac{2}{5}} = 2.076$.
28. $\dfrac{(-.1412)^2}{\sqrt[3]{-(.00475)}} = -.11858$.
29. $\dfrac{1}{(72.32)^{\frac{2}{3}}} = .05761$.

TABLE II.

This table gives the logarithms of the sine, cosine, tangent and cotangent of angles from 0° to 90°, at intervals of 10′.

When the angle is taken from the left-hand colum of the page, the name of the function must be sought at the top of the page; when the angle is taken from the right-hand column of the page, the name of the function must be sought at the foot of the page.

When the function is numerically less than 1, − 10 must be written after its tabular logarithm. This is the case with the sines and cosines of all angles between 0° and 90°, with tangents of angles between 0° and 45°, and with cotangents between 45° and 90°.

For convenience in interpolation the differences of the tabular logarithms are given, and these differences are subdivided into tenths in the column of proportional parts. Hence this column contains the corrections to the tabular logarithms for each minute of angle from 1′ to 9′ inclusive. These corrections are to be added when the logarithm increases with the angle, and they are to be subtracted when the logarithm decreases as the angle increases.

When the logarithm of a function of an angle greater than 90° is required, change to the equivalent function of an angle less than 90° (§21). Algebraic signs must be adjusted independently of the logarithmic calculation, as in the use of Table I.

Seconds of arc must be reduced to the equivalent fractions of a minute of arc.

To obtain log sec x, take from the table colog cos x; for log csc x use colog sin x.

Examples.

1. $\log \sin 20° \, 13' = ?$

$$\begin{array}{rll} \log \sin 20° \, 10' = & 9.5375; & d = 34. \\ d \text{ for } 3' \text{ (Prop. Parts)} = & \underline{\quad 10.2} & \\ \log \sin 20° \, 13' = & 9.5385 - 10. & \end{array}$$

2. $\log \cos 20° \, 13' = ?$

$$\begin{array}{rll} \log \cos 20° \, 10' = & 9.9725; & d = 4. \\ d \text{ for } 3' = 4 \times .3 = & \underline{\quad 1.2} & \\ \log \cos 20° \, 13' = & 9.9724 - 10. & \end{array}$$

3. $\log \tan 29^\circ\ 47' = ?$

$$\begin{aligned} \log \tan 29^\circ\ 40' &= 9.7556; \qquad d = 29. \\ d \text{ for } 7' \text{ (Prop. Parts)} &= \underline{20.3} \\ \log \tan 29^\circ\ 47' &= 9.7576 - 10 \end{aligned}$$

The same result may also be obtained by starting with log tan 29° 50′, thus:

$$\begin{aligned} \log \tan 29^\circ\ 50' &= 9.7585; \qquad d = 29. \\ d \text{ for } 3' &= \underline{8.7} \\ \log \tan 29^\circ\ 47' &= 9.7576 - 10. \end{aligned}$$

As a rule, in interpolating start from the nearest tabular number.

4. $\log \cot 29^\circ\ 47' = ?$

$$\begin{aligned} \log \cot 29^\circ\ 50' &= 0.2415; \qquad d = 29. \\ d \text{ for } 3' &= \underline{8.7} \\ \log \cot 29^\circ\ 47' &= 0.2424. \end{aligned}$$

5. $\log \sin 58^\circ\ 44' = ?$

$$\begin{aligned} \log \sin 58^\circ\ 40' &= 9.9315; \qquad d = 8. \\ d \text{ for } 4' &= \underline{3.2} \\ \log \sin 58^\circ\ 44' &= 9.9318 - 10. \end{aligned}$$

6. $\log \tan 67^\circ\ 23.5' = ?$

$$\begin{aligned} \log \tan 67^\circ\ 20' &= 0.3792; \qquad d = 36. \\ d \text{ for } 3.5' = 10.8 + 1.8 &= \underline{12.6} \\ \log \tan 67^\circ\ 23.5' &= 0.3805. \end{aligned}$$

Here we obtain d for 3.5′ from d for 3′ + d for 0.5′. Note that d for 0.5 is simply one-tenth of d for 5′.

7. $\log \cos 105^\circ\ 51.6' = ?$

$$\cos 105^\circ\ 51.6' = -\sin 15^\circ\ 51.6'.$$

Neglecting the algebraic sign we have

$$\begin{aligned} \log \sin 15^\circ\ 50' &= 9.4359; \qquad d = 44. \\ d \text{ for } 1.6' &= \underline{7.0} \\ \log \sin 15^\circ\ 51.6' &= 9.4366 - 10 = \log \cos 105^\circ\ 51.6'. \end{aligned}$$

8. $\log \tan 250^\circ\ 34.3' = ?$

$$\begin{aligned} \tan 250^\circ\ 34.3' &= \tan 70^\circ\ 34.3'. \\ \log \tan 70^\circ\ 30' &= 0.4509; \qquad d = 40. \\ d \text{ for } 4.3' &= \underline{17.2} \\ \log \tan 70^\circ\ 34.3' &= 0.4526 = \log \tan 250^\circ\ 34.3'. \end{aligned}$$

Angles near 0° or near 90°.

When an angle, x, lies near 0°, sin x, tan x, and cot x vary too rapidly with x to permit of accurate interpolation of their loga-

rithms from the table. The same is true of cos x, tan x, and cot x, when x lies near 90°. We will show how accurate values of these logarithms may be obtained.

Let $$S = \log \frac{\sin x}{x} \quad \text{and} \quad T = \log \frac{\tan x}{x},$$

x being expressed in minutes of arc. We indicate this by x'.

Then $$\log \sin x = \log x' + S,$$
and $$\log \tan x = \log x' + T.$$

When x is small the quantities S and T vary quite slowly with x. The values of S and T are given in the last column of the first page of Table II, x ranging from 0° to 5°; -10 is to be added to the tabular numbers there given.

To get log sin x, reduce x to minutes of arc and take log x' from Table I; to this logarithm add S.

To get log tan x, add T to log x'.

To get log cot x, first get log tan x and form the cologarithm of the result.

For, $$\log \cot x = \operatorname{colog} \tan x.$$

To obtain log cos x, log tan x or log cot x, when x lies between 85° and 90°, calculate the co-function of the complementary angle by the method given above.

To find the angle from log sin x, log tan x or log cot x, when x lies near 0°, we use the relations

$$\log x' = \log \sin x - S;$$
$$\log x' = \log \tan x - T;$$
$$\log x' = -\log \cot x - T.$$

The necessary values of S and T can be obtained after finding an approximate value of x from Table II.

To find x from log cos x, log tan x, or log cot x, when x lies near 90°, replace

$$\log \cos x \quad \text{by} \quad \log \sin (90° - x);$$
$$\log \tan x \quad \text{by} \quad \log \cot (90° - x);$$
$$\log \cot x \quad \text{by} \quad \log \tan (90° - x).$$

Then $90° - x$ can be obtained by the method given above for angles near 0°. Hence x is determined.

Examples.

1. Find log sin x, log tan x and log cot x when $x = 1° \, 22' \, 12''$.

$x = 1° \, 22' \, 12'' = 82.2'$. $\qquad$ $\log x' = \log 82.2 = 1.9149$.

$$\begin{array}{rlrl} \log x &= 1.9149 & \log x &= 1.9149 \\ S &= \underline{6.4637 - 10} & T &= \underline{6.4638 - 10} \\ \log \sin x &= 8.3786 - 10 & \log \tan x &= 8.3787 - 10 \end{array}$$

$$\log \cot x = \text{colog} \tan x = 1.6213.$$

2. Find log cos x, log tan x and log cot x when $x = 89° \, 5' \, 50''$.

Let $\qquad y = 90° - x = 54' \, 10'' = 54.17'$.

Then log cos x, log tan x, log cot x are equal respectively to log sin y, log cot y, log tan y, which may be found as in example 1.

3. $\qquad \log \sin x = 8.2142; \qquad x = ?$

From Table II, $x = 50' +$; hence $S = 6.4637 - 10$.

$$\begin{array}{rl} \log \sin x &= 8.2142 - 10 \\ S &= \underline{6.4637 - 10} \\ \log x' &= 1.7505; \qquad x = 56.30' = 56' \, 18''. \end{array}$$

4. $\qquad \log \tan x = 8.0804 - 10; \qquad x = ?$

From Table II, $x = 40' +$; hence $T = 6.4638$

$$\begin{array}{rl} \log \tan x &= 8.0804 - 10 \\ T &= \underline{6.4638 - 10} \\ \log x' &= 1.6166; \qquad x = 41.36' = 41' \, 21.6''. \end{array}$$

5. $\qquad \log \cot x = 8.6276 - 10; \qquad x = ?$

Let $\qquad y = 90° - x$.

Then $\qquad \log \tan y = \log \cot x = 8.6276 - 10$.

From Table II, $y = 2° \, 20' +$; hence $T = 6.4640$.

$$\begin{array}{rl} \log \tan y &= 8.6276 - 10 \\ T &= \underline{6.4640 - 10} \\ \log y' &= 2.1636; \quad y = 145.73' = 2° \, 25' \, 44''. \end{array}$$

Hence $\qquad x = 90° - y = 87° \, 34' \, 16''$.

Let the student obtain the results required in the last five examples by direct interpolation from Table II.

Exercises. Verify the following equations:

1. log sin $20° \, 40' = 9.5477 - 10$.
2. log cos $66° \, 30' = 9.6007 - 10$.
3. log tan $29° \, 35' = 9.7541 - 10$.
4. log cot $37° \, 25' = 0.1163$.
5. log sec $55° \, 50' = 0.2506$.
6. log csc $44° \, 50' = 0.1518$.
7. log tan $63° \, 27' = 0.3013$.
8. log sin $81° \, 29' = 9.9952$.
9. log sin $81° \, 31' = 9.9952$.
10. log cos $81° \, 29' = 9.1706 - 10$.
11. log cos $81° \, 31' = 9.1689 - 10$.
12. log cot $9° \, 6' = 0.7954$.

13. $\log \sin 152° 27' = 9.6651 - 10$. **16.** $\log \cot 0° 10' 22'' = 2.5206$.
14. $\log \sin 2° 10' 10'' = 8.5781 - 10$. **17.** $\log \cos 89° 28' 44'' = 7.9588 - 10$.
15. $\log \tan 1° 34' 20'' = 8.4385 - 10$. **18.** $\log \tan 88° 46' 14'' = 1.6683$.
19. $\log \sin x = 9.7926$; $x = 38° 20'$.
20. $\log \sin x = 9.3548$; $x = 13° 5'$.
21. $\log \sin x = 9.8867$; $x = 50° 23'$.
22. $\log \cos x = 9.6030$; $x = 66° 22'$.
23. $\log \tan x = 0.6278$; $x = 77° 44.5'$.
24. $\log \cot x = 0.0906$; $x = 39° 4'$.
25. $\log \cot x = 0.6648$; $x = 12° 12.5'$.
26. $\log \sec x = 0.1374$; $x = 43° 13'$.
27. $\log \csc x = 0.2890$; $x = 30° 56'$.
28. $\log \sec x = 0.6680$; $x = 77° 35.8'$.
29. $\log \sin x = 8.3698$; $x = 1° 20' 34''$.
30. $\log \tan x = 8.7659$; $x = 3° 20' 18''$.
31. $\log \cot x = 1.2952$; $x = 2° 54' 3''$.
32. $\log \cos x = 8.5387$; $x = 88° 1' 8''$.
33. $\log \cot x = 7.9485$; $x = 89° 29' 28''$.
34. $\log \csc x = 2.3549$; $x = 0° 15' 11''$.
35. $\log \sec x = 1.5102$; $x = 88° 13' 48''$.

TABLE III

This table gives the numerical values of the six trigonometric functions of angles from 0° to 90° intervals at of 10′. The functions of intermediate angles are to be obtained by interpolation.

By using the tables inversely, an angle may be found, usually to the nearest minute, when a function of the angle is known to four decimal places.

TABLE IV

A 4-place table of natural and logarithmic haversines at intervals of 10′ from 0° to 180°.

TABLE V

This is a conversion table for changing from sexagesimal to radian measure, and conversely. The entries are given to five decimal places in radians, corresponding nearly to 2″ in sexagesimal measure.

Examples.

1. Express 200° 44′ 36″ in radian measure.

$$200° = 3 \times 60° + 20°$$

$3 \times 60° = 3 \times 1.04720 =$	3.14160 radians.	
$20° =$	0.34907	
$44' =$	0.01280	
$36'' =$	0.00017	
$200°\ 44'\ 36''$	= 3.50364 radians.	

2. Express 3.50364 radians in sexagesimal measure.

3.0	radians	=	171° 53′ 14″
0.5	"	=	28° 38′ 52″
0.003	"	=	10′ 19″
0.0006	"	=	2′ 4″
0.00004	"	=	8″
3.50364	radians	=	200° 44′ 37″

TABLE VI

This table contains the values of a number of mathematical constants, generally to fifteen places of decimals.

TABLE VII

This table gives the values of the natural or Naperian logarithm of x, and of the ascending and decending exponential functions e^x and e^{-x}, from $x = 0$ to $x = 5$ at intervals of 0.05. As a rule the tabular entries are given to three decimal places.

TABLE VIII

This table gives the values of n^2, n^3, $\sqrt{n}$, and $\sqrt[3]{n}$, for values of n from 1 to 100.

The direct use of the table requires no explanation. As an example of its inverse use we find the approximate value of $\sqrt[3]{320}$. We have

$$(6.8)^3 = 314.432 \quad (n = 68),$$
$$(6.9)^3 = 328.509 \quad (n = 69).$$

Hence, interpolating linearly,

$$(6.840)^3 = 320 \text{ approx., or } \sqrt[3]{320} = 6.840+.$$

TABLES

I. Logarithms of Numbers

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	43	**42**
1	4.3	4.2
2	8.6	8.4
3	12.9	12.6
4	17.2	16.8
5	21.5	21.0
6	25.8	25.2
7	30.1	29.4
8	34.4	33.6
9	38.7	37.8
	41	**40**
1	4.1	4.0
2	8.2	8.0
3	12.3	12.0
4	16.4	16.0
5	20.5	20.0
6	24.6	24.0
7	28.7	28.0
8	32.8	32.0
9	36.9	36.0
	39	**38**
1	3.9	3.8
2	7.8	7.6
3	11.7	11.4
4	15.6	15.2
5	19.5	19.0
6	23.4	22.8
7	27 3	26.6
8	31.2	30.4
9	35.1	34.2
	37	**36**
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4
	35	**34**
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
5	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6
	33	**32**
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
6	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

Prop. Parts

I. Logarithms of Numbers

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	31	**30**
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

	29	**28**
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	**26**
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25	**24**
1	2.5	2.4
2	5.0	4.8
3	7.5	7.2
4	10.0	9.6
5	12.5	12.0
6	15.0	14.4
7	17.5	16.8
8	20.0	19.2
9	22.5	21.6

	23	**22**
1	2.3	2.2
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8

	21
1	2.1
2	4.2
3	6.3
4	8.4
5	10.5
6	12.6
7	14.7
8	16.8
9	18.9

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
0° 0′	−∞		10.0000	0	−∞		∞	90° 0′
10′	7.4637	3011	.0000	0	7.4637	3011	2.5363	50′
20′	.7648	1760	.0000	0	.7648	1761	.2352	40′
30′	.9408	1250	.0000	0	.9409	1249	.0591	30′
40′	8.0658	969	.0000	0	8.0658	969	1.9342	20′
50′	.1627	792	.0000	1	.1627	792	.8373	10′
1° 0′	8.2419	669	9.9999	0	8.2419	670	1.7581	89° 0′
10′	.3088	580	.9999	0	.3089	580	.6911	50′
20′	.3668	511	.9999	0	.3669	512	.6331	40′
30′	.4179	458	9.9999	1	.4181	457	.5819	30′
40′	.4637	413	.9998	0	.4638	415	.5362	20′
50′	.5050	378	.9998	1	.5053	378	.4947	10′
2° 0′	8.5428	348	9.9997	0	8.5431	348	1.4569	88° 0′
10′	.5776	321	.9997	1	.5779	322	.4221	50′
20′	.6097	300	.9996	0	.6101	300	.3899	40′
30′	.6397	280	.9996	1	.6401	281	.3599	30′
40′	.6677	263	.9995	0	.6682	263	.3318	20′
50′	.6940	248	.9995	1	.6945	249	.3055	10′
3° 0′	8.7188	235	9.9994	1	8.7194	235	1.2806	87° 0′
10′	.7423	222	.9993	0	.7429	223	.2571	50′
20′	.7645	212	.9993	1	.7652	213	.2348	40′
30′	.7857	202	.9992	1	.7865	202	.2135	30′
40′	.8059	192	.9991	1	.8067	194	.1933	20′
50′	.8251	185	.9990	1	.8261	185	.1739	10′
4° 0′	8.8436	177	9.9989	0	8.8446	178	1.1554	86° 0′
10′	.8613	170	.9989	1	.8624	171	.1376	50′
20′	.8783	163	.9988	1	.8795	165	.1205	40′
30′	.8946	158	.9987	1	.8960	158	.1040	30′
40′	.9104	152	.9986	1	.9118	154	.0882	20′
50′	.9256	147	.9985	2	.9272	148	.0728	10′
5° 0′	8.9403	142	9.9983	1	8.9420	143	1.0580	85° 0′
10′	.9545	137	.9982	1	.9563	138	.0437	50′
20′	.9682	134	.9981	1	.9701	135	.0299	40′
30′	.9816	129	.9980	1	.9836	130	.0164	30′
40′	.9945	125	.9979	2	.9966	127	.0034	20′
50′	9.0070	122	.9977	1	9.0093	123	0.9907	10′
6° 0′	9.0192	119	9.9976	1	9.0216	120	0.9784	84° 0′
10′	.0311	115	.9975	2	.0336	117	.9664	50′
20′	.0426	113	.9973	1	.0453	114	.9547	40′
30′	.0539	109	.9972	1	.0567	111	.9433	30′
40′	.0648	107	.9971	2	.0678	108	.9322	20′
50′	.0755	104	.9969	1	.0786	105	.9214	10′
7° 0′	9.0859	102	9.9968	2	9.0891	104	0.9109	83° 0′
10′	.0961	99	.9966	2	.0995	101	.9005	50′
20′	.1060	97	.9964	1	.1096	98	.8904	40′
30′	.1157		.9963		.1194		.8806	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Small Angles

x	S	T
<1°	6.4637	6.4637
1°	6.4637	6.4638
2°	6.4636	6.4639
3°	6.4635	6.4641
4°	6.4634	6.4644
5°	6.4631	6.4649

Prop. Parts.

	113	111	109
1	11.3	11.1	10.9
2	22.6	22.2	21.8
3	33.9	33.3	32.7
4	45.2	44.4	43.6
5	56.5	55.5	54.5
6	67.8	66.6	65.4
7	79.1	77.7	76.3
8	90.4	88.8	87.2
9	101.7	99.9	98.1

	108	107	105
1	10.8	10.7	10.5
2	21.6	21.4	21.0
3	32.4	32.1	31.5
4	43.2	42.8	42.0
5	54.0	53.5	52.5
6	64.8	64.2	63.0
7	75.6	74.9	73.5
8	86.4	85.6	84.0
9	97.2	96.3	94.5

	104	102	101
1	10.4	10.2	10.1
2	20.8	20.4	20.2
3	31.2	30.6	30.3
4	41.6	40.8	40.4
5	52.0	51.0	50.5
6	62.4	61.2	60.6
7	72.8	71.4	70.7
8	83.2	81.6	80.8
9	93.6	91.8	90.9

	99	98	97	95
1	9.9	9.8	9.7	9.5
2	19.8	19.6	19.4	19.0
3	29.7	29.4	29.1	28.5
4	39.6	39.2	38.8	38.0
5	49.5	49.0	48.5	47.5
6	59.4	58.8	58.2	57.0
7	69.3	68.6	67.9	66.5
8	79.2	78.4	77.6	76.0
9	89.1	88.2	87.3	85.5

	143	142	138	137	135	134	130	129	127	125	123	122	119	117	115	114
1	14.3	14.2	13.8	13.7	13.5	13.4	13.0	12.9	12.7	12.5	12.3	12.2	11.9	11.7	11.5	11.4
2	28.6	28.4	27.6	27.4	27.0	26.8	26.0	25.8	25.4	25.0	24.6	24.4	23.8	23.4	23.0	22.8
3	42.9	42.6	41.4	41.1	40.5	40.2	39.0	38.7	38.1	37.5	36.9	36.6	35.7	35.1	34.5	34.2
4	57.2	56.8	55.2	54.8	54.0	53.6	52.0	51.6	50.8	50.0	49.2	48.8	47.6	46.8	46.0	45.6
5	71.5	71.0	69.0	68.5	67.5	67.0	65.0	64.5	63.5	62.5	61.5	61.0	59.5	58.5	57.5	57.0
6	85.8	85.2	82.8	82.2	81.0	80.4	78.0	77.4	76.2	75.0	73.8	73.2	71.4	70.2	69.0	68.4
7	100.1	99.4	96.6	95.9	94.5	93.8	91.0	90.3	88.9	87.5	86.1	85.4	83.3	81.9	80.5	79.8
8	114.4	113.6	110.4	109.6	108.0	107.2	104.0	103.2	101.6	100.0	98.4	97.6	95.2	93.6	92.0	91.2
9	128.7	127.8	124.2	123.3	121.5	120.6	117.0	116.1	114.3	112.5	110.7	109.8	107.1	105.3	103.5	102.6

II. Logarithms of Trigonometric Functions

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.1157	95	9.9963	2	9.1194	97	0.8806	30′
40′	.1252	93	.9961	2	.1291	94	.8709	20′
50′	.1345	91	.9959	1	.1385	93	.8615	10′
8° 0′	9.1436	89	9.9958	2	9.1478	91	0.8522	**82° 0′**
10′	.1525	87	.9956	2	.1569	89	.8431	50′
20′	.1612	85	.9954	2	.1658	87	.8342	40′
30′	.1697	84	.9952	2	.1745	86	.8255	30′
40′	.1781	82	.9950	2	.1831	84	.8169	20′
50′	.1863	80	.9948	2	.1915	82	.8085	10′
9° 0′	9.1943	79	9.9946	2	9.1997	81	0.8003	**81° 0′**
10′	.2022	78	.9944	2	.2078	80	.7922	50′
20′	.2100	76	.9942	2	.2158	78	.7842	40′
30′	.2176	75	.9940	2	.2236	77	.7764	30′
40′	.2251	73	.9938	2	.2313	76	.7687	20′
50′	.2324	73	.9936	2	.2389	74	.7611	10′
10° 0′	9.2397	71	9.9934	3	9.2463	73	0.7537	**80° 0′**
10′	.2468	70	.9931	2	.2536	73	.7464	50′
20′	.2538	68	.9929	2	.2609	71	.7391	40′
30′	.2606	68	.9927	3	.2680	70	.7320	30′
40′	.2674	66	.9924	2	.2750	69	.7250	20′
50′	.2740	66	.9922	3	.2819	68	.7181	10′
11° 0′	9.2806	64	9.9919	2	9.2887	66	0.7113	**79° 0′**
10′	.2870	64	.9917	3	.2953	67	.7047	50′
20′	.2934	63	.9914	2	.3020	65	.6980	40′
30′	.2997	61	.9912	3	.3085	64	.6915	30′
40′	.3058	61	.9909	2	.3149	63	.6851	20′
50′	.3119	60	.9907	3	.3212	63	.6788	10′
12° 0′	9.3179	59	9.9904	3	9.3275	61	0.6725	**78° 0′**
10′	.3238	58	.9901	2	.3336	61	.6664	50′
20′	.3296	57	.9899	3	.3397	61	.6603	40′
30′	.3353	57	.9896	3	.3458	59	.6542	30′
40′	.3410	56	.9893	3	.3517	59	.6483	20′
50′	.3466	55	.9890	3	.3576	58	.6424	10′
13° 0′	9.3521	54	9.9887	3	9.3634	57	0.6366	**77° 0′**
10′	.3575	54	.9884	3	.3691	57	.6309	50′
20′	.3629	53	.9881	3	.3748	56	.6252	40′
30′	.3682	52	.9878	3	.3804	55	.6196	30′
40′	.3734	52	.9875	3	.3859	55	.6141	20′
50′	.3786	51	.9872	3	.3914	54	.6086	10′
14° 0′	9.3837	50	9.9869	3	9.3968	53	0.6032	**76° 0′**
10′	.3887	50	.9866	3	.4021	53	.5979	50′
20′	.3937	49	.9863	4	.4074	53	.5926	40′
30′	.3986	49	.9859	3	.4127	51	.5873	30′
40′	.4035	48	.9856	3	.4178	52	.5822	20′
50′	.4083	47	.9853	4	.4230	51	.5770	10′
15° 0′	9.4130		9.9849		9.4281		0.5719	**75° 0′**
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	73	71	70	69	68
1	7.3	7.1	7.0	6.9	6.8
2	14.6	14.2	14.0	13.8	13.6
3	21.9	21.3	21.0	20.7	20.4
4	29.2	28.4	28.0	27.6	27.2
5	36.5	35.5	35.0	34.5	34.0
6	43.8	42.6	42.0	41.4	40.8
7	51.1	49.7	49.0	48.3	47.6
8	58.4	56.8	56.0	55.2	54.4
9	65.7	63.9	63.0	62.1	61.2

	67	66	65	64	63
1	6.7	6.6	6.5	6.4	6.3
2	13.4	13.2	13.0	12.8	12.6
3	20.1	19.8	19.5	19.2	18.9
4	26.8	26.4	26.0	25.6	25.2
5	33.5	33.0	32.5	32.0	31.5
6	40.2	39.6	39.0	38.4	37.8
7	46.9	46.2	45.5	44.8	44.1
8	53.6	52.8	52.0	51.2	50.4
9	60.3	59.4	58.5	57.6	56.7

	61	60	59	58	57
1	6.1	6.0	5.9	5.8	5.7
2	12.2	12.0	11.8	11.6	11.4
3	18.3	18.0	17.7	17.4	17.1
4	24.4	24.0	23.6	23.2	22.8
5	30.5	30.0	29.5	29.0	28.5
6	36.6	36.0	35.4	34.8	34.2
7	42.7	42.0	41.3	40.6	39.9
8	48.8	48.0	47.2	46.4	45.6
9	54.9	54.0	53.1	52.2	51.3

	56	55	54	53	52
1	5.6	5.5	5.4	5.3	5.2
2	11.2	11.0	10.8	10.6	10.4
3	16.8	16.5	16.2	15.9	15.6
4	22.4	22.0	21.6	21.2	20.8
5	28.0	27.5	27.0	26.5	26.0
6	33.6	33.0	32.4	31.8	31.2
7	39.2	38.5	37.8	37.1	36.4
8	44.8	44.0	43.2	42.4	41.6
9	50.4	49.5	48.6	47.7	46.8

	51	50	48	47
1	5.1	5.0	4.8	4.7
2	10.2	10.0	9.6	9.4
3	15.3	15.0	14.4	14.1
4	20.4	20.0	19.2	18.8
5	25.5	25.0	24.0	23.5
6	30.6	30.0	28.8	28.2
7	35.7	35.0	33.6	32.9
8	40.8	40.0	38.4	37.6
9	45.9	45.5	43.2	42.3

	97	94	93	91	89	87	86	85	84	82	81	79	78	77	76	75	74
1	9.7	9.4	9.3	9.1	8.9	8.7	8.6	8.5	8.4	8.2	8.1	7.9	7.8	7.7	7.6	7.5	7.4
2	19.4	18.8	18.6	18.2	17.8	17.4	17.2	17.0	16.8	16.4	16.2	15.8	15.6	15.4	15.2	15.0	14.8
3	29.1	28.2	27.9	27.3	26.7	26.1	25.8	25.5	25.2	24.6	24.3	23.7	23.4	23.1	22.8	22.5	22.2
4	38.8	37.6	37.2	36.4	35.6	34.8	34.4	34.0	33.6	32.8	32.4	31.6	31.2	30.8	30.4	30.0	29.6
5	48.5	47.0	46.5	45.5	44.5	43.5	43.0	42.5	42.0	41.0	40.5	39.5	39.0	38.5	38.0	37.5	37.0
6	58.2	56.4	55.8	54.6	53.4	52.2	51.6	51.0	50.4	49.2	48.6	47.4	46.8	46.2	45.6	45.0	44.4
7	67.9	65.8	65.1	63.7	62.3	60.9	60.2	59.5	58.8	57.4	56.7	55.3	54.6	53.9	53.2	52.5	51.8
8	77.6	75.2	74.4	72.8	71.2	69.6	68.8	68.0	67.2	65.6	64.8	63.2	62.4	61.6	60.8	60.0	59.2
9	87.3	84.6	83.7	81.9	80.1	78.3	77.4	76.5	75.6	73.8	72.9	71.1	70.2	69.3	68.4	67.5	66.6

x	log sin	d	log cos	d	log tan	d	log cot	
15° 0′	9.4130	47	9.9849	3	9.4281	50	0.5719	75° 0′
10′	.4177	46	.9846	3	.4331	50	.5669	50′
20′	.4223	46	.9843	4	.4381	49	.5619	40′
30′	.4269	45	.9839	3	.4430	49	.5570	30′
40′	.4314	45	.9836	4	.4479	48	.5521	20′
50′	.4359	44	.9832	4	.4527	48	.5473	10′
16° 0′	9.4403	44	9.9828	3	9.4575	47	0.5425	74° 0′
10′	.4447	44	.9825	4	.4622	47	.5378	50′
20′	.4491	42	.9821	4	.4669	47	.5331	40′
30′	.4533	43	.9817	3	.4716	46	.5284	30′
40′	.4576	42	.9814	4	.4762	46	.5238	20′
50′	.4618	41	.9810	4	.4808	45	.5192	10′
17° 0′	9.4659	41	9.9806	4	9.4853	45	0.5147	73° 0′
10′	.4700	41	.9802	4	.4898	45	.5102	50′
20′	.4741	40	.9798	4	.4943	44	.5057	40′
30′	.4781	40	.9794	4	.4987	44	.5013	30′
40′	.4821	40	.9790	4	.5031	44	.4969	20′
50′	.4861	39	.9786	4	.5075	43	.4925	10′
18° 0′	9.4900	39	9.9782	4	9.5118	43	0.4882	72° 0′
10′	.4939	38	.9778	4	.5161	42	.4839	50′
20′	.4977	38	.9774	4	.5203	42	.4797	40′
30′	.5015	37	.9770	5	.5245	42	.4755	30′
40′	.5052	38	.9765	4	.5287	42	.4713	20′
50′	.5090	36	.9761	4	.5329	41	.4671	10′
19° 0′	9.5126	37	9.9757	5	9.5370	41	0.4630	71° 0′
10′	.5163	36	.9752	4	.5411	40	.4589	50′
20′	.5199	36	.9748	5	.5451	40	.4549	40′
30′	.5235	35	.9743	4	.5491	40	.4509	30′
40′	.5270	36	.9739	5	.5531	40	.4469	20′
50′	.5306	35	.9734	4	.5571	40	.4429	10′
20° 0′	9.5341	34	9.9730	5	9.5611	39	0.4389	70° 0′
10′	.5375	34	.9725	4	.5650	39	.4350	50′
20′	.5409	34	.9721	5	.5689	38	.4311	40′
30′	.5443	34	.9716	5	.5727	39	.4273	30′
40′	.5477	33	.9711	5	.5766	38	.4234	20′
50′	.5510	33	.9706	4	.5804	38	.4196	10′
21° 0′	9.5543	33	9.9702	5	9.5342	37	0.4158	69° 0′
10′	.5576	33	.9697	5	.5879	38	.4121	50′
20′	.5609	32	.9692	5	.5917	37	.4083	40′
30′	.5641	32	.9687	5	.5954	37	.4046	30′
40′	.5673	31	.9682	5	.5991	37	.4009	20′
50′	.5704	32	.9677	5	.6028	36	.3972	10′
22° 0′	9.5736	31	9.9672	5	9.6064	36	0.3936	68° 0′
10′	.5767	31	.9667	6	.6100	36	.3900	50′
20′	.5798	30	.9661	5	.6136	36	.3864	40′
30′	.5828		.9656		.6172		.3828	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	50	49	48	47
1	5.0	4.9	4.8	4.7
2	10.0	9.8	9.6	9.4
3	15.0	14.7	14.4	14.1
4	20.0	19.6	19.2	18.8
5	25.0	24.5	24.0	23.5
6	30.0	29.4	28.8	28.2
7	35.0	34.3	33.6	32.9
8	40.0	39.2	38.4	37.6
9	45.0	44.1	43.2	42.3

	46	45	44	43
1	4.6	4.5	4.4	4.3
2	9.2	9.0	8.8	8.6
3	13.8	13.5	13.2	12.9
4	18.4	18.0	17.6	17.2
5	23.0	22.5	22.0	21.5
6	27.6	27.0	26.4	25.8
7	32.2	31.5	30.8	30.1
8	36.8	36.0	35.2	34.4
9	41.4	40.5	39.6	38.7

	42	41	40	39
1	4.2	4.1	4.0	3.9
2	8.4	8.2	8.0	7.8
3	12.6	12.3	12.0	11.7
4	16.8	16.4	16.0	15.6
5	21.0	20.5	20.0	19.5
6	25.2	24.6	24.0	23.4
7	29.4	28.7	28.0	27.3
8	33.6	32.8	32.0	31.2
9	37.8	36.9	36.0	35.1

	38	37	36	35
1	3.8	3.7	3.6	3.5
2	7.6	7.4	7.2	7.0
3	11.4	11.1	10.8	10.5
4	15.2	14.8	14.4	14.0
5	19.0	18.5	18.0	17.5
6	22.8	22.2	21.6	21.0
7	26.6	25.9	25.2	24.5
8	30.4	29.6	28.8	28.0
9	34.2	33.3	32.4	31.5

	34	33	32	31
1	3.4	3.3	3.2	3.1
2	6.8	6.6	6.4	6.2
3	10.2	9.9	9.6	9.3
4	13.6	13.2	12.8	12.4
5	17.0	16.5	16.0	15.5
6	20.4	19.8	19.2	18.6
7	23.8	23.1	22.4	21.7
8	27.2	26.4	25.6	24.8
9	30.6	29.7	28.8	27.9

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.5828	31	9.9656	5	9.6172	36	0.3828	30′
40′	.5859	30	.9651	5	.6208	35	.3792	20′
50′	.5889	30	.9646	6	.6243	36	.3757	10′
23° 0′	9.5919	29	9.9640	5	9.6279	35	0.3721	67° 0′
10′	.5948	30	.9635	6	.6314	34	.3686	50′
20′	.5978	29	.9629	5	.6348	35	.3652	40′
30′	.6007	29	.9624	6	.6383	34	.3617	30′
40′	.6036	29	.9618	5	.6417	35	.3583	20′
50′	.6065	28	.9613	6	.6452	34	.3548	10′
24° 0′	9.6093	28	9.9607	5	9.6486	34	0.3514	66° 0′
10′	.6121	28	.9602	6	.6520	33	.3480	50′
20′	.6149	28	.9596	6	.6553	34	.3447	40′
30′	.6177	28	.9590	6	.6587	33	.3413	30′
40′	.6205	27	.9584	5	.6620	34	.3380	20′
50′	.6232	27	.9579	6	.6654	33	.3346	10′
25° 0′	9.6259	27	9.9573	6	9.6687	33	0.3313	65° 0′
10′	.6286	27	.9567	6	.6720	32	.3280	50′
20′	.6313	27	.9561	6	.6752	33	.3248	40′
30′	.6340	26	.9555	6	.6785	32	.3215	30′
40′	.6366	26	.9549	6	.6817	33	.3183	20′
50′	.6392	26	.9543	6	.6850	32	.3150	10′
26° 0′	9.6418	26	9.9537	7	9.6882	32	0.3118	64° 0′
10′	.6444	26	.9530	6	.6914	32	.3086	50′
20′	.6470	25	.9524	6	.6946	31	.3054	40′
30′	.6495	26	.9518	6	.6977	32	.3023	30′
40′	.6521	25	.9512	7	.7009	31	.2991	20′
50′	.6546	24	.9505	6	.7040	32	.2960	10′
27° 0′	9.6570	25	9.9499	7	9.7072	31	0.2928	63° 0′
10′	.6595	25	.9492	6	.7103	31	.2897	50′
20′	.6620	24	.9486	7	.7134	31	.2866	40′
30′	.6644	24	.9479	6	.7165	31	.2835	30′
40′	.6668	24	.9473	7	.7196	30	.2804	20′
50′	.6692	24	.9466	7	.7226	31	.2774	10′
28° 0′	9.6716	24	9.9459	6	9.7257	30	0.2743	62° 0′
10′	.6740	23	.9453	7	.7287	30	.2713	50′
20′	.6763	24	.9446	7	.7317	31	.2683	40′
30′	.6787	23	.9439	7	.7348	30	.2652	30′
40′	.6810	23	.9432	7	.7378	30	.2622	20′
50′	.6833	23	.9425	7	.7408	30	.2592	10′
29° 0′	9.6856	22	9.9418	7	9.7438	29	0.2562	61° 0′
10′	.6878	23	.9411	7	.7467	30	.2533	50′
20′	.6901	22	.9404	7	.7497	29	.2503	40′
30′	.6923	23	.9397	7	.7526	30	.2474	30′
40′	.6946	22	.9390	7	.7556	29	.2444	20′
50′	.6968	22	.9383	8	.7585	29	.2415	10′
30° 0′	9.6990		9.9375		9.7614		0.2386	60° 0′
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	36	35	34
1	3.6	3.5	3.4
2	7.2	7.0	6.8
3	10.8	10.5	10.2
4	14.4	14.0	13.6
5	18.0	17.5	17.0
6	21.6	21.0	20.4
7	25.2	24.5	23.8
8	28.8	28.0	27.2
9	32.4	31.5	30.6

	33	32	31
1	3.3	3.2	3.1
2	6.6	6.4	6.2
3	9.9	9.6	9.3
4	13.2	12.8	12.4
5	16.5	16.0	15.5
6	19.8	19.2	18.6
7	23.1	22.4	21.7
8	26.4	25.6	24.8
9	29.7	28.8	27.9

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	27	26	25
1	2.7	2.6	2.5
2	5.4	5.2	5.0
3	8.1	7.8	7.5
4	10.8	10.4	10.0
5	13.5	13.0	12.5
6	16.2	15.6	15.0
7	18.9	18.2	17.5
8	21.6	20.8	20.0
9	24.3	23.4	22.5

	24	23	22
1	2.4	2.3	2.2
2	4.8	4.6	4.4
3	7.2	6.9	6.6
4	9.6	9.2	8.8
5	12.0	11.5	11.0
6	14.4	13.8	13.2
7	16.8	16.1	15.4
8	19.2	18.4	17.6
9	21.6	20.7	19.8

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30° 0′	9.6990	22	9.9375	7	9.7614	30	0.2386	60° 0′
10′	.7012	21	.9368	7	.7644	29	.2356	50′
20′	.7033	22	.9361	8	.7673	28	.2327	40′
30′	.7055	21	.9353	7	.7701	29	.2299	30′
40′	.7076	21	.9346	8	.7730	29	.2270	20′
50′	.7097	21	.9338	7	.7759	29	.2241	10′
31° 0′	9.7118	21	9.9331	8	9.7788	28	0.2212	59° 0′
10′	.7139	21	.9323	8	.7816	29	.2184	50′
20′	.7160	21	.9315	7	.7845	28	.2155	40′
30′	.7181	20	.9308	8	.7873	29	.2127	30′
40′	.7201	21	.9300	8	.7902	28	.2098	20′
50′	.7222	20	.9292	8	.7930	28	.2070	10′
32° 0′	9.7242	20	9.9284	8	9.7958	28	0.2042	58° 0′
10′	.7262	20	.9276	8	.7986	28	.2014	50′
20′	.7282	20	.9268	8	.8014	28	.1986	40′
30′	.7302	20	.9260	8	.8042	28	.1958	30′
40′	.7322	20	.9252	8	.8070	27	.1930	20′
50′	.7342	19	.9244	8	.8097	28	.1903	10′
33° 0′	9.7361	19	9.9236	8	9.8125	28	0.1875	57° 0′
10′	.7380	20	.9228	9	.8153	27	.1847	50′
20′	.7400	19	.9219	8	.8180	28	.1820	40′
30′	.7419	19	.9211	8	.8208	27	.1792	30′
40′	.7438	19	.9203	9	.8235	28	.1765	20′
50′	.7457	19	.9194	8	.8263	27	.1737	10′
34° 0′	9.7476	18	9.9186	9	9.8290	27	0.1710	56° 0′
10′	.7494	19	.9177	8	.8317	27	.1683	50′
20′	.7513	18	.9169	9	.8344	27	.1656	40′
30′	.7531	19	.9160	9	.8371	27	.1629	30′
40′	.7550	18	.9151	9	.8398	27	.1602	20′
50′	.7568	18	.9142	8	.8425	27	.1575	10′
35° 0′	9.7586	18	9.9134	9	9.8452	27	0.1548	55° 0′
10′	.7604	18	.9125	9	.8479	27	.1521	50′
20′	.7622	18	.9116	9	.8506	27	.1494	40′
30′	.7640	17	.9107	9	.8533	26	.1467	30′
40′	.7657	18	.9098	9	.8559	27	.1441	20′
50′	.7675	17	.9089	9	.8586	27	.1414	10′
36° 0′	9.7692	18	9.9080	10	9.8613	26	0.1387	54° 0′
10′	.7710	17	.9070	9	.8639	27	.1361	50′
20′	.7727	17	.9061	9	.8666	26	.1334	40′
30′	.7744	17	.9052	10	.8692	26	.1308	30′
40′	.7761	17	.9042	9	.8718	27	.1282	20′
50′	.7778	17	.9033	10	.8745	26	.1255	10′
37° 0′	9.7795	16	9.9023	9	9.8771	26	0.1229	53° 0′
10′	.7811	17	.9014	10	.8797	27	.1203	50′
20′	.7828	16	.9004	9	.8824	26	.1176	40′
30′	.7844		.8995		.8850		.1150	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	27	26	22
1	2.7	2.6	2.2
2	5.4	5.2	4.4
3	8.1	7.8	6.6
4	10.8	10.4	8.8
5	13.5	13.0	11.0
6	16.2	15.6	13.2
7	18.9	18.2	15.4
8	21.6	20.8	17.6
9	24.3	23.4	19.8

	21	20	19
1	2.1	2.0	1.9
2	4.2	4.0	3.8
3	6.3	6.0	5.7
4	8.4	8.0	7.6
5	10.5	10.0	9.5
6	12.6	12.0	11.4
7	14.7	14.0	13.3
8	16.8	16.0	15.2
9	18.9	18.0	17.1

	18	17	16
1	1.8	1.7	1.6
2	3.6	3.4	3.2
3	5.4	5.1	4.8
4	7.2	6.8	6.4
5	9.0	8.5	8.0
6	10.8	10.2	9.6
7	12.6	11.9	11.2
8	14.4	13.6	12.8
9	16.2	15.3	14.4

	9	8	7
1	.9	.8	.7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.7844	17	9.8995	10	9.8850	26	0.1150	30′
40′	.7861	16	.8985	10	.8876	26	.1124	20′
50′	.7877	16	.8975	10	.8902	26	.1098	10′
38° 0′	9.7893	17	9.8965	10	9.8928	26	0.1072	**52° 0′**
10′	.7910	16	.8955	10	.8954	26	.1046	50′
20′	.7926	15	.8945	10	.8980	26	.1020	40′
30′	.7941	16	.8935	10	.9006	26	.0994	30′
40′	.7957	16	.8925	10	.9032	26	.0968	20′
50′	.7973	16	.8915	10	.9058	26	.0942	10′
39° 0′	9.7989	15	9.8905	10	9.9084	26	0.0916	**51° 0′**
10′	.8004	16	.8895	11	.9110	25	.0890	50′
20′	.8020	15	.8884	10	.9135	26	.0865	40′
30′	.8035	15	.8874	10	.9161	26	.0839	30′
40′	.8050	16	.8864	11	.9187	25	.0813	20′
50′	.8066	15	.8853	10	.9212	26	.0788	10′
40° 0′	9.8081	15	9.8843	11	9.9238	26	0.0762	**50° 0′**
10′	.8096	15	.8832	11	.9264	25	.0736	50′
20′	.8111	14	.8821	11	.9289	26	.0711	40′
30′	.8125	15	.8810	10	.9315	26	.0685	30′
40′	.8140	15	.8800	11	.9341	25	.0659	20′
50′	.8155	14	.8789	11	.9366	26	.0634	10′
41° 0′	9.8169	15	9.8778	11	9.9392	25	0.0608	**49° 0′**
10′	.8184	14	.8767	11	.9417	26	.0583	50′
20′	.8198	15	.8756	11	.9443	25	.0557	40′
30′	.8213	14	.8745	12	.9468	26	.0532	30′
40′	.8227	14	.8733	11	.9494	25	.0506	20′
50′	.8241	14	.8722	11	.9519	25	.0481	10′
42° 0′	9.8255	14	9.8711	12	9.9544	26	0.0456	**48° 0′**
10′	.8269	14	.8699	11	.9570	25	.0430	50′
20′	.8283	14	.8688	12	.9595	26	.0405	40′
30′	.8297	14	.8676	11	.9621	25	.0379	30′
40′	.8311	13	.8665	12	.9646	25	.0354	20′
50′	.8324	14	.8653	12	.9671	26	.0329	10′
43° 0′	9.8338	13	9.8641	12	9.9697	25	0.0303	**47° 0′**
10′	.8351	14	.8629	11	.9722	25	.0278	50′
20′	.8365	13	.8618	12	.9747	25	.0253	40′
30′	.8378	13	.8606	12	.9772	26	.0228	30′
40′	.8391	14	.8594	12	.9798	25	.0202	20′
50′	.8405	13	.8582	13	.9823	25	.0177	10′
44° 0′	9.8418	13	9.8569	12	9.9848	26	0.0152	**46° 0′**
10′	.8431	13	.8557	12	.9874	25	.0126	50′
20′	.8444	13	.8545	13	.9899	25	.0101	40′
30′	.8457	12	.8532	12	.9924	25	.0076	30′
40′	.8469	13	.8520	13	.9949	26	.0051	20′
50′	.8482	13	.8507	12	.9975	25	.0025	10′
45° 0′	9.8495		9.8495		0.0000		0.0000	**45° 0′**
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	17	16	15
1	1.7	1.6	1.5
2	3.4	3.2	3.0
3	5.1	4.8	4.5
4	6.8	6.4	6.0
5	8.5	8.0	7.5
6	10.2	9.6	9.0
7	11.9	11.2	10.5
8	13.6	12.8	12.0
9	15.3	14.4	13.5

	14	13	12
1	1.4	1.3	1.2
2	2.8	2.6	2.4
3	4.2	3.9	3.6
4	5.6	5.2	4.8
5	7.0	6.5	6.0
6	8.4	7.8	7.2
7	9.8	9.1	8.4
8	11.2	10.4	9.6
9	12.6	11.7	10.8

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

Prop. Parts

III. Natural Values of Trigonometric Functions

x	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$	
0° 0′	.00000	1.0000	.00000	∞	1.0000	∞	90° 0′
10′	.00291	1.0000	.00291	343.77	1.0000	343.78	50′
20′	.00582	1.0000	.00582	171.88	1.0000	171.89	40′
30′	.00873	1.0000	.00873	114.59	1.0000	114.59	30′
40′	.01164	.9999	.01164	85.940	1.0001	85.946	20′
50′	.01454	.9999	.01455	68.750	1.0001	68.757	10′
1° 0′	.01745	.9998	.01746	57.290	1.0002	57.299	89° 0′
10′	.02036	.9998	.02036	49.104	1.0002	49.114	50′
20′	.02327	.9997	.02328	42.964	1.0003	42.976	40′
30′	.02618	.9997	.02619	38.188	1.0003	38.202	30′
40′	.02908	.9996	.02910	34.368	1.0004	34.382	20′
50′	.03199	.9995	.03201	31.242	1.0005	31.258	10′
2° 0′	.03490	.9994	.03492	28.6363	1.0006	28.654	88° 0′
10′	.03781	.9993	.03783	26.4316	1.0007	26.451	50′
20′	.04071	.9992	.04075	24.5418	1.0008	24.562	40′
30′	.04362	.9990	.04366	22.9038	1.0010	22.926	30′
40′	.04653	.9989	.04658	21.4704	1.0011	21.494	20′
50′	.04943	.9988	.04949	20.2056	1.0012	20.230	10′
3° 0′	.05234	.9986	.05241	19.0811	1.0014	19.107	87° 0′
10′	.05524	.9985	.05533	18.0750	1.0015	18.103	50′
20′	.05814	.9983	.05824	17.1693	1.0017	17.198	40′
30′	.06105	.9981	.06116	16.3499	1.0019	16.380	30′
40′	.06395	.9980	.06408	15.6048	1.0021	15.637	20′
50′	.06685	.9978	.06700	14.9244	1.0022	14.958	10′
4° 0′	.06976	.9976	.06993	14.3007	1.0024	14.336	86° 0′
10′	.07266	.9974	.07285	13.7267	1.0027	13.763	50′
20′	.07556	.9971	.07578	13.1969	1.0029	13.235	40′
30′	.07846	.9969	.07870	12.7062	1.0031	12.746	30′
40′	.08136	.9967	.08163	12.2505	1.0033	12.291	20′
50′	.08426	.9964	.08456	11.8262	1.0036	11.868	10′
5° 0′	.08716	.9962	.08749	11.4301	1.0038	11.474	85° 0′
10′	.09005	.9959	.09042	11.0594	1.0041	11.105	50′
20′	.09295	.9957	.09335	10.7119	1.0044	10.758	40′
30′	.09585	.9954	.09629	10.3854	1.0046	10.433	30′
40′	.09874	.9951	.09923	10.0780	1.0049	10.128	20′
50′	.10164	.9948	.10216	9.7882	1.0052	9.839	10′
6° 0′	.10453	.9945	.10510	9.5144	1.0055	9.5668	84° 0′
10′	.10742	.9942	.10805	9.2553	1.0058	9.3092	50′
20′	.11031	.9939	.11099	9.0098	1.0061	9.0652	40′
30′	.11320	.9936	.11394	8.7769	1.0065	8.8337	30′
40′	.11609	.9932	.11688	8.5555	1.0068	8.6138	20′
50′	.11898	.9929	.11983	8.3450	1.0072	8.4647	10′
7° 0′	.12187	.9925	.12278	8.1443	1.0075	8.2055	83° 0′
10′	.12476	.9922	.12574	7.9530	1.0079	8.0157	50′
20′	.12764	.9918	.12869	7.7704	1.0083	7.8344	40′
30′	.13053	.9914	.13165	7.5958	1.0086	7.6613	30′
	$\cos x$	$\sin x$	$\cot x$	$\tan x$	$\operatorname{cosec} x$	$\sec x$	x

III. Natural Values of Trigonometric Functions

x	sin x	cos x	tan x	cot x	sec x	cosec x	
30′	.1305	.9914	.1317	7.5958	1.0086	7.6613	30′
40′	.1334	.9911	.1346	7.4287	1.0090	7.4957	20′
50′	.1363	.9907	.1376	7.2687	1.0094	7.3372	10′
8° 0′	.1392	.9903	.1405	7.1154	1.0098	7.1853	82° 0′
10′	.1421	.9899	.1435	6.9682	1.0102	7.0396	50′
20′	.1449	.9894	.1465	6.8269	1.0107	6.8998	40′
30′	.1478	.9890	.1495	6.6912	1.0111	6.7655	30′
40′	.1507	.9886	.1524	6.5606	1.0116	6.6363	20′
50′	.1536	.9881	.1554	6.4348	1.0120	6.5121	10′
9° 0′	.1564	.9877	.1584	6.3138	1.0125	6.3925	81° 0′
10′	.1593	.9872	.1614	6.1970	1.0129	6.2772	50′
20′	.1622	.9868	.1644	6.0844	1.0134	6.1661	40′
30′	.1650	.9863	.1673	5.9758	1.0139	6.0589	30′
40′	.1679	.9858	.1703	5.8708	1.0144	5.9554	20′
50′	.1708	.9853	.1733	5.7694	1.0149	5.8554	10′
10° 0′	.1736	.9848	.1763	5.6713	1.0154	5.7588	80° 0′
10′	.1765	.9843	.1793	5.5764	1.0160	5.6653	50′
20′	.1794	.9838	.1823	5.4845	1.0165	5.5749	40′
30′	.1822	.9833	.1853	5.3955	1.0170	5.4874	30′
40′	.1851	.9827	.1883	5.3093	1.0176	5.4026	20′
50′	.1880	.9822	.1914	5.2257	1.0182	5.3205	10′
11° 0′	.1908	.9816	.1944	5.1446	1.0187	5.2408	79° 0′
10′	.1937	.9811	.1974	5.0658	1.0193	5.1636	50′
20′	.1965	.9805	.2004	4.9894	1.0199	5.0886	40′
30′	.1994	.9799	.2035	4.9152	1.0205	5.0159	30′
40′	.2022	.9793	.2065	4.8430	1.0211	4.9452	20′
50′	.2051	.9787	.2095	4.7729	1.0217	4.8765	10′
12° 0′	.2079	.9781	.2126	4.7046	1.0223	4.8097	78° 0′
10′	.2108	.9775	.2156	4.6382	1.0230	4.7448	50′
20′	.2136	.9769	.2186	4.5736	1.0236	4.6817	40′
30′	.2164	.9763	.2217	4.5107	1.0243	4.6202	30′
40′	.2193	.9757	.2247	4.4494	1.0249	4.5604	20′
50′	.2221	.9750	.2278	4.3897	1.0256	4.5022	10′
13° 0′	.2250	.9744	.2309	4.3315	1.0263	4.4454	77° 0′
10′	.2278	.9737	.2339	4.2747	1.0270	4.3901	50′
20′	.2306	.9730	.2370	4.2193	1.0277	4.3362	40′
30′	.2334	.9724	.2401	4.1653	1.0284	4.2837	30′
40′	.2363	.9717	.2432	4.1126	1.0291	4.2324	20′
50′	.2391	.9710	.2462	4.0611	1.0299	4.1824	10′
14° 0′	.2419	.9703	.2493	4.0108	1.0306	4.1336	76° 0′
10′	.2447	.9696	.2524	3.9617	1.0314	4.0859	50′
20′	.2476	.9689	.2555	3.9136	1.0321	4.0394	40′
30′	.2504	.9681	.2586	3.8667	1.0329	3.9939	30′
40′	.2532	.9674	.2617	3.8208	1.0337	3.9495	20′
50′	.2560	.9667	.2648	3.7760	1.0345	3.9061	10′
15° 0′	.2588	.9659	.2679	3.7321	1.0353	3.8637	75° 0′
	cos x	sin x	cot x	tan x	cosec x	sec x	x

III. Natural Values of Trigonometric Functions

x	sin x	cos x	tan x	cot x	sec x	cosec x	
15° 0′	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75° 0′**
10′	.2616	.9652	.2711	3.6891	1.0361	3.8222	50′
20′	.2644	.9644	.2742	3.6470	1.0369	3.7817	40′
30′	.2672	.9636	.2773	3.6059	1.0377	3.7420	30′
40′	.2700	.9628	.2805	3.5656	1.0386	3.7032	20′
50′	.2728	.9621	.2836	3.5261	1.0394	3.6652	10′
16° 0′	.2756	.9613	.2867	3.4874	1.0403	3.6280	**74° 0′**
10′	.2784	.9605	.2899	3.4495	1.0412	3.5915	50′
20′	.2812	.9596	.2931	3.4124	1.0421	3.5559	40′
30′	.2840	.9588	.2962	3.3759	1.0430	3.5209	30′
40′	.2868	.9580	.2994	3.3402	1.0439	3.4867	20′
50′	.2896	.9572	.3026	3.3052	1.0448	3.4532	10′
17° 0′	.2924	.9563	.3057	3.2709	1.0457	3.4203	**73° 0′**
10′	.2952	.9555	.3089	3.2371	1.0466	3.3881	50′
20′	.2979	.9546	.3121	3.2041	1.0476	3.3565	40′
30′	.3007	.9537	.3153	3.1716	1.0485	3.3255	30′
40′	.3035	.9528	.3185	3.1397	1.0495	3.2951	20′
50′	.3062	.9520	.3217	3.1084	1.0505	3.2653	10′
18° 0′	.3090	.9511	.3249	3.0777	1.0515	3.2361	**72° 0′**
10′	.3118	.9502	.3281	3.0475	1.0525	3.2074	50′
20′	.3145	.9492	3314	3.0178	1.0535	3.1792	40′
30′	.3173	.9483	.3346	2.9887	1.0545	3.1516	30′
40′	.3201	.9474	.3378	2.9600	1.0555	3.1244	20′
50′	.3228	.9465	.3411	2.9319	1.0566	3.0977	10′
19° 0′	.3256	.9455	.3443	2.9042	1.0576	3.0716	**71° 0′**
10′	.3283	.9446	.3476	2.8770	1.0587	3.0458	50′
20′	.3311	.9436	.3508	2.8502	1.0598	3.0206	40′
30′	.3338	.9426	.3541	2.8239	1.0609	2.9957	30′
40′	.3365	.9417	.3574	2.7980	1.0620	2.9714	20′
50′	.3393	.9407	.3607	2.7725	1.0631	2.9474	10′
20° 0′	.3420	.9397	.3640	2.7475	1.0642	2.9238	**70° 0′**
10′	.3448	.9387	.3673	2.7228	1.0653	2.9006	50′
20′	.3475	.9377	.3706	2.6985	1.0665	2.8779	40′
30′	.3502	.9367	.3739	2.6746	1.0676	2.8555	30′
40′	.3529	.9356	.3772	2.6511	1.0688	2.8334	20′
50′	.3557	.9346	.3805	2.6279	1.0700	2.8118	10′
21° 0′	.3584	.9336	.3839	2.6051	1.0712	2.7904	**69° 0′**
10′	.3611	.9325	.3872	2.5826	1.0724	2.7695	50′
20′	.3638	.9315	.3906	2.5605	1.0736	2.7488	40′
30′	.3665	.9304	.3939	2.5386	1.0748	2.7285	30′
40′	.3692	.9293	.3973	2.5172	1.0760	2.7085	20′
50′	.3719	.9283	.4006	2.4960	1.0773	2.6888	10′
22° 0′	.3746	.9272	.4040	2.4751	1.0785	2.6695	**68° 0′**
10′	.3773	.9261	.4074	2.4545	1.0798	2.6504	50′
20′	.3800	.9250	.4108	2.4342	1.0811	2.6316	40′
30′	.3827	.9239	.4142	2.4142	1.0824	2 6131	30′
	cos x	sin x	cot x	tan x	cosec x	sec x	x

III. Natural Values of Trigonometric Functions

x		sin x	cos x	tan x	cot x	sec x	cosec x		
	30′	.3827	.9239	.4142	2.4142	1.0824	2.6131		30′
	40′	.3854	.9228	.4176	2.3945	1.0837	2.5949		20′
	50′	.3881	.9216	.4210	2.3750	1.0850	2.5770		10′
23°	**0′**	.3907	.9205	.4245	2.3559	1.0864	2.5593	**67°**	**0′**
	10′	.3934	.9194	.4279	2.3369	1.0877	2.5419		50′
	20′	.3961	.9182	.4314	2.3183	1.0891	2.5247		40′
	30′	.3987	.9171	.4348	2.2998	1.0904	2.5078		30′
	40′	.4014	.9159	.4383	2.2817	1.0918	2.4912		20′
	50′	.4041	.9147	.4417	2.2637	1.0932	2.4748		10′
24°	**0′**	.4067	.9135	.4452	2.2460	1.0946	2.4586	**66°**	**0′**
	10′	.4094	.9124	.4487	2.2286	1.0961	2.4426		50′
	20′	.4120	.9112	.4522	2.2113	1.0975	2.4269		40′
	30′	.4147	.9100	.4557	2.1943	1.0990	2.4114		30′
	40′	.4173	.9088	.4592	2.1775	1.1004	2.3961		20′
	50′	.4200	.9075	.4628	2.1609	1.1019	2.3811		10′
25°	**0′**	.4226	.9063	.4663	2.1445	1.1034	2.3662	**65°**	**0′**
	10′	.4253	.9051	.4699	2.1283	1.1049	2.3515		50′
	20′	.4279	.9038	.4734	2.1123	1.1064	2.3371		40′
	30′	.4305	.9026	.4770	2.0965	1.1079	2.3228		30′
	40′	.4331	.9013	.4806	2.0809	1.1095	2.3088		20′
	50′	.4358	.9001	.4841	2.0655	1.1110	2.2949		10′
26°	**0′**	.4384	.8988	.4877	2.0503	1.1126	2.2812	**64°**	**0′**
	10′	.4410	.8975	.4913	2.0353	1.1142	2.2677		50′
	20′	.4436	.8962	.4950	2.0204	1.1158	2.2543		40′
	30′	.4462	.8949	.4986	2.0057	1.1174	2.2412		30′
	40′	.4488	.8936	.5022	1.9912	1.1190	2.2282		20′
	50′	.4514	.8923	.5059	1.9768	1.1207	2.2154		10′
27°	**0′**	.4540	.8910	.5095	1.9626	1.1223	2.2027	**63°**	**0′**
	10′	.4566	.8897	.5132	1.9486	1.1240	2.1902		50′
	20′	.4592	.8884	.5169	1.9347	1.1257	2.1779		40′
	30′	.4617	.8870	.5206	1.9210	1.1274	2.1657		30′
	40′	.4643	.8857	.5243	1.9074	1.1291	2.1537		20′
	50′	.4669	.8843	.5280	1.8940	1.1308	2.1418		10′
28°	**0′**	.4695	.8829	.5317	1.8807	1.1326	2.1301	**62°**	**0′**
	10′	.4720	.8816	.5354	1.8676	1.1343	2.1185		50′
	20′	.4746	.8802	.5392	1.8546	1.1361	2.1070		40′
	30′	.4772	.8788	.5430	1.8418	1.1379	2.0957		30′
	40′	.4797	.8774	.5467	1.8291	1.1397	2.0846		20′
	50′	.4823	.8760	.5505	1.8165	1.1415	2.0736		10′
29°	**0′**	.4848	.8746	.5543	1.8040	1.1434	2.0627	**61°**	**0′**
	10′	.4874	.8732	.5581	1.7917	1.1452	2.0519		50′
	20′	.4899	.8718	.5619	1.7796	1.1471	2.0413		40′
	30′	.4924	.8704	.5658	1.7675	1.1490	2.0308		30′
	40′	.4950	.8689	.5696	1.7556	1.1509	2.0204		20′
	50′	.4975	.8675	.5735	1.7437	1.1528	2.0101		10′
30°	**0′**	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60°**	**0′**
		cos x	sin x	cot x	tan x	cosec x	sec x	x	

III. Natural Values of Trigonometric Functions

x	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$	
30° 0′	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60°** 0′
10′	.5025	.8646	.5812	1.7205	1.1567	1.9900	50′
20′	.5050	.8631	.5851	1.7090	1.1586	1.9801	40′
30′	.5075	.8616	.5890	1.6977	1.1606	1.9703	30′
40′	.5100	.8601	.5930	1.6864	1.1626	1.9606	20′
50′	.5125	.8587	.5969	1.6753	1.1646	1.9511	10′
31° 0′	.5150	.8572	.6009	1.6643	1.1666	1.9416	**59°** 0′
10′	.5175	.8557	.6048	1.6534	1.1687	1.9323	50′
20′	.5200	.8542	.6088	1.6426	1.1708	1.9230	40′
30′	.5225	.8526	.6128	1.6319	1.1728	1.9139	30′
40′	.5250	.8511	.6168	1.6212	1.1749	1.9049	20′
50′	.5275	.8496	.6208	1.6107	1.1770	1.8959	10′
32° 0′	.5299	.8480	.6249	1.6003	1.1792	1.8871	**58°** 0′
10′	.5324	.8465	.6289	1.5900	1.1813	1.8783	50′
20′	.5348	.8450	.6330	1.5798	1.1835	1.8699	40′
30′	.5373	.8434	.6371	1.5697	1.1857	1.8612	30′
40′	.5398	.8418	.6412	1.5597	1.1879	1.8527	20′
50′	.5422	.8403	.6453	1.5497	1.1901	1.8444	10′
33° 0′	.5446	.8387	.6494	1.5399	1.1924	1.8361	**57°** 0′
10′	.5471	.8371	.6536	1.5301	1.1946	1.8279	50′
20′	.5495	.8355	.6577	1.5204	1.1969	1.8198	40′
30′	.5519	.8339	.6619	1.5108	1.1992	1.8118	30′
40′	.5544	.8323	.6661	1.5013	1.2015	1.8039	20′
50′	.5568	.8307	.6703	1.4919	1.2039	1 7960	10′
34° 0′	.5592	.8290	.6745	1.4826	1.2062	1.7883	**56°** 0′
10′	.5616	.8274	.6787	1.4733	1.2086	1.7806	50′
20′	.5640	.8258	.6830	1.4641	1.2110	1.7730	40′
30′	.5664	.8241	.6873	1.4550	1.2134	1.7655	30′
40′	.5688	.8225	.6916	1.4460	1.2158	1.7581	20′
50′	.5712	.8208	.6959	1.4370	1.2183	1.7507	10′
35° 0′	.5736	.8192	.7002	1.4281	1.2208	1.7435	**55°** 0′
10′	.5760	.8175	.7046	1.4193	1.2233	1.7362	50′
20′	.5783	.8158	.7089	1.4106	1.2258	1.7291	40′
30′	.5807	.8141	.7133	1.4019	1.2283	1.7221	30′
40′	.5831	.8124	.7177	1.3934	1.2309	1.7151	20′
50′	.5854	.8107	.7221	1.3848	1.2335	1.7082	10′
36° 0′	.5878	.8090	.7265	1.3764	1.2361	1.7013	**54°** 0′
10′	.5901	.8073	.7310	1.3680	1.2387	1.6945	50′
20′	.5925	.8056	.7355	1.3597	1.2413	1.6878	40′
30′	.5948	.8039	.7400	1.3514	1.2440	1.6812	30′
40′	.5972	.8021	.7445	1.3432	1.2467	1.6746	20′
50′	.5995	.8004	.7490	1.3351	1.2494	1.6681	10′
37° 0′	.6018	.7986	.7536	1.3270	1.2521	1.6616	**53°** 0′
10′	.6041	.7969	.7581	1.3190	1.2549	1.6553	50′
20′	.6065	.7951	.7627	1.3111	1.2577	1.6489	40′
30′	.6088	.7934	.7673	1.3032	1.2605	1.6427	30′
	$\cos x$	$\sin x$	$\cot x$	$\tan x$	$\operatorname{cosec} x$	$\sec x$	x

III. Natural Values of Trigonometric Functions

x	sin x	cos x	tan x	cot x	sec x	cosec x	
30′	.6088	.7934	.7673	1.3032	1.2605	1.6427	30′
40′	.6111	.7916	.7720	1.2954	1.2633	1.6365	20′
50′	.6134	.7898	.7766	1.2876	1.2662	1.6304	10′
38° 0′	.6157	.7880	.7813	1.2799	1.2690	1.6243	**52° 0′**
10′	.6180	.7862	.7860	1.2723	1.2719	1.6183	50′
20′	.6202	.7844	.7907	1.2647	1.2748	1.6123	40′
30′	.6225	.7826	.7954	1.2572	1.2779	1.6064	30′
40′	.6248	.7808	.8002	1.2497	1.2808	1.6005	20′
50′	.6271	.7790	.8050	1.2423	1.2837	1.5948	10′
39° 0′	.6293	.7771	.8098	1.2349	1.2868	1.5890	**51° 0′**
10′	.6316	.7753	.8146	1.2276	1.2898	1.5833	50′
20′	.6338	.7735	.8195	1.2203	1.2929	1.5777	40′
30′	.6361	.7716	.8243	1.2131	1.2960	1.5721	30′
40′	.6383	.7698	.8292	1.2059	1.2991	1.5666	20′
50′	.6406	.7679	.8342	1.1988	1.3022	1.5611	10′
40° 0′	.6428	.7660	.8391	1.1918	1.3054	1.5557	**50° 0′**
10′	.6450	.7642	.8441	1.1847	1.3086	1.5504	50′
20′	.6472	.7623	.8491	1.1778	1.3118	1.5450	40′
30′	.6494	.7604	.8541	1.1708	1 3151	1.5398	30′
40′	.6517	.7585	.8591	1.1640	1.3184	1.5346	20′
50′	.6539	.7566	.8642	1.1571	1.3217	1.5294	10′
41° 0′	.6561	.7547	.8693	1.1504	1.3250	1.5243	**49° 0′**
10′	.6583	.7528	.8744	1.1436	1.3284	1.5192	50′
20′	.6604	.7509	.8796	1.1369	1.3318	1.5142	40′
30′	.6626	.7490	.8847	1.1303	1.3352	1.5092	30′
40′	.6648	.7470	.8899	1.1237	1.3386	1.5042	20′
50′	.6670	.7451	.8952	1.1171	1.3421	1.4993	10′
42° 0′	.6691	.7431	.9004	1.1106	1.3456	1.4945	**48° 0′**
10′	.6713	.7412	.9057	1.1041	1.3492	1.4897	50′
20′	.6734	.7392	.9110	1.0977	1.3527	1.4849	40′
30′	.6756	.7373	.9163	1.0913	1.3563	1.4802	30′
40′	.6777	.7353	.9217	1.0850	1.3600	1.4755	20′
50′	.6799	.7333	.9271	1.0786	1.3636	1.4709	10′
43° 0′	.6820	.7314	.9325	1.0724	1.3673	1.4663	**47° 0′**
10′	.6841	.7294	.9380	1.0661	1.3711	1.4617	50′
20′	.6862	.7274	.9435	1.0599	1.3748	1.4572	40′
30′	.6884	.7254	.9490	1.0538	1.3786	1.4527	30′
40′	.6905	.7234	.9545	1.0477	1.3824	1.4483	20′
50′	.6926	.7214	.9601	1.0416	1.3863	1.4439	10′
44° 0′	.6947	.7193	.9657	1.0355	1.3902	1.4396	**46° 0′**
10′	.6967	.7173	.9713	1.0295	1.3941	1.4352	50′
20′	.6988	.7153	.9770	1.0235	1.3980	1.4310	40′
30′	.7009	.7133	.9827	1.0176	1.4020	1.4267	30′
40′	.7030	.7112	.9884	1.0117	1.4061	1.4225	20′
50′	.7050	.7092	.9942	1.0058	1.4101	1.4184	10′
45° 0′	.7071	.7071	1.0000	1.0000	1.4142	1.4142	**45° 0′**
	cos x	sin x	cot x	tan x	cosec x	sec x	x

IV. Haversines, Natural and Logarithmic

	Nat.	Log.		Nat.	Log.		Nat.	Log.
0° 0′	.0000	—∞	**8° 0′**	.0049	7.6872	**16° 0′**	.0194	8.2871
10′	.0000	4.3254	10′	.0051	7.7050	10′	.0198	8.2961
20′	.0000	4.9275	20′	.0053	7.7226	20′	.0202	8.3049
30′	.0000	5.2796	30′	.0055	7.7397	30′	.0206	8.3137
40′	.0000	5.5295	40′	.0057	7.7566	40′	.0210	8.3223
50′	.0001	5.7233	50′	.0059	7.7731	50′	.0214	8.3309
1° 0′	.0001	5.8817	**9° 0′**	.0062	7.7893	**17° 0′**	.0218	8.3394
10′	.0001	6.0156	10′	.0064	7.8052	10′	.0223	8.3478
20′	.0001	6.1315	20′	.0066	7.8208	20′	.0227	8.3561
30′	.0002	6.2388	30′	.0069	7.8361	30′	.0231	8.3644
40′	.0002	6.3254	40′	.0071	7.8512	40′	.0236	8.3726
50′	.0003	6.4081	50′	.0073	7.8660	50′	.0240	8.3806
2° 0′	.0003	6.4837	**10° 0′**	.0076	7.8806	**18° 0′**	.0245	8.3887
10′	.0004	6.5532	10′	.0079	7.8949	10′	.0249	8.3966
20′	.0004	6.6176	20′	.0081	7.9090	20′	.0254	8.4045
30′	.0005	6.6775	30′	.0084	7.9229	30′	.0258	8.4123
40′	.0005	6.7336	40′	.0086	7.9365	40′	.0263	8.4200
50′	.0006	6.7862	50′	.0089	7.9499	50′	.0268	8.4276
3° 0′	.0007	6.8358	**11° 0′**	.0092	7.9631	**19° 0′**	.0272	8.4352
10′	.0008	6.8828	10′	.0095	7.9762	10′	.0277	8.4427
20′	.0008	6.9273	20′	.0097	7.9890	20′	.0282	8.4502
30′	.0009	6.9697	30′	.0100	8.0016	30′	.0287	8.4576
40′	.0010	7.0101	40′	.0103	8.0141	40′	.0292	8.4649
50′	.0011	7.0487	50′	.0106	8.0264	50′	.0297	8.4721
4° 0′	.0012	7.0856	**12° 0′**	.0109	8.0385	**20° 0′**	.0302	8.4793
10′	.0013	7.1211	10′	.0112	8.0504	10′	.0307	8.4865
20′	.0014	7.1551	20′	.0115	8.0622	20′	.0312	8.4935
30′	.0015	7.1879	30′	.0119	8.0738	30′	.0317	8.5006
40′	.0017	7.2195	40′	.0122	8.0852	40′	.0322	8.5075
50′	.0018	7.2499	50′	.0125	8.0966	50′	.0327	8.5144
5° 0′	.0019	7.2794	**13° 0′**	.0128	8.1077	**21° 0′**	.0332	8.5213
10′	.0020	7.3078	10′	.0131	8.1187	10′	.0337	8.5281
20′	.0022	7.3354	20′	.0135	8.1296	20′	.0343	8.5348
30′	.0023	7.3621	30′	.0138	8.1404	30′	.0348	8.5415
40′	.0024	7.3880	40′	.0142	8.1510	40′	.0353	8.5481
50′	.0026	7.4132	50′	.0145	8.1614	50′	.0359	8.5547
6° 0′	.0027	7.4376	**14° 0′**	.0149	8.1718	**22° 0′**	.0364	8.5612
10′	.0029	7.4614	10′	.0152	8.1820	10′	.0370	8.5677
20′	.0031	7.4845	20′	.0156	8.1921	20′	.0375	8.5741
30′	.0032	7.5071	30′	.0159	8.2021	30′	.0381	8.5805
40′	.0034	7.5290	40′	.0163	8.2120	40′	.0386	8.5868
50′	.0036	7.5504	50′	.0167	8.2217	50′	.0392	8.5931
7° 0′	.0037	7.5713	**15° 0′**	.0170	8.2314	**23° 0′**	.0397	8.5993
10′	.0039	7.5918	10′	.0174	8.2409	10′	.0403	8.6055
20′	.0041	7.6117	20′	.0178	8.2504	20′	.0409	8.6116
30′	.0043	7.6312	30′	.0182	8.2597	30′	.0415	8.6177
40′	.0045	7.6503	40′	.0186	8.2689	40′	.0421	8.6238
50′	.0047	7.6689	50′	.0190	8.2781	50′	.0426	8.6298

IV. Haversines, Natural and Logarithmic

	Nat.	Log.		Nat.	Log.		Nat.	Log.
24° 0′	.0432	8.6358	**32° 0′**	.0760	8.8807	**40° 0′**	.1170	9.0681
10′	.0438	8.6417	10′	.0767	8.8851	10′	.1179	9.0716
20′	.0444	8.6476	20′	.0775	8.8894	20′	.1189	9.0750
30′	.0450	8.6534	30′	.0783	8.8938	30′	.1198	9.0784
40′	.0456	8.6592	40′	.0791	8.8981	40′	.1207	9.0819
50′	.0462	8.6650	50′	.0799	8.9024	50′	.1217	9.0853
25° 0′	.0468	8.6707	**33° 0′**	.0807	8.9067	**41° 0′**	.1226	9.0887
10′	.0475	8.6764	10′	.0815	8.9109	10′	.1236	9.0920
20′	.0481	8.6820	20′	.0823	8.9152	20′	.1246	9.0954
30′	.0487	8.6876	30′	.0831	8.9194	30′	.1255	9.0987
40′	.0493	8.6932	40′	.0839	8.9236	40′	.1265	9.1020
50′	.0500	8.6987	50′	.0847	8.9277	50′	.1275	9.1054
26° 0′	.0506	8.7042	**34° 0′**	.0855	8.9319	**42° 0′**	.1284	9.1087
10′	.0512	8.7096	10′	.0863	8.9360	10′	.1294	9.1119
20′	.0519	8.7150	20′	.0871	8.9401	20′	.1304	9.1152
30′	.0525	8.7204	30′	.0879	8.9442	30′	.1314	9.1185
40′	.0532	8.7258	40′	.0888	8.9482	40′	.1323	9.1217
50′	.0538	8.7311	50′	.0896	8.9523	50′	.1333	9.1249
27° 0′	.0545	8.7364	**35° 0′**	.0904	8.9563	**43° 0′**	.1343	9.1281
10′	.0552	8.7416	10′	.0913	8.9603	10′	.1353	9.1314
20′	.0558	8.7468	20′	.0921	8.9643	20′	.1363	9.1345
30′	.0565	8.7520	30′	.0929	8.9682	30′	.1373	9.1377
40′	.0572	8.7572	40′	.0938	8.9721	40′	.1383	9.1409
50′	.0578	8.7623	50′	.0946	8.9761	50′	.1393	9.1440
28° 0′	.0585	8.7673	**36° 0′**	.0955	8.9800	**44° 0′**	.1403	9.1472
10′	.0592	8.7724	10′	.0963	8.9838	10′	.1413	9.1503
20′	.0599	8.7774	20′	.0972	8.9877	20′	.1424	9.1534
30′	.0606	8.7824	30′	.0981	8.9915	30′	1434	9.1565
40′	.0613	8.7874	40′	.0989	8.9954	40′	.1444	9.1596
50′	.0620	8.7923	50′	.0998	8.9992	50′	.1454	9.1626
29° 0′	.0627	8.7972	**37° 0′**	.1007	9.0030	**45° 0′**	.1464	9.1657
10′	.0634	8.8021	10′	.1016	9.0067	10′	.1475	9.1687
20′	.0641	8.8069	20′	.1024	9.0105	20′	.1485	9.1718
30′	.0648	8.8117	30′	.1033	9.0142	30′	.1495	9.1748
40′	.0655	8.8165	40′	.1042	9.0179	40′	.1506	9.1778
50′	.0663	8.8213	50′	.1051	9.0216	50′	.1516	9.1808
30° 0′	.0670	8.8260	**38° 0′**	.1060	9.0253	**46° 0′**	.1527	9.1838
10′	.0677	8.8307	10′	.1069	9.0289	10′	.1537	9.1867
20′	.0684	8.8354	20′	.1078	9.0326	20′	.1548	9.1897
30′	.0692	8.8400	30′	.1087	9.0362	30′	.1558	9.1926
40′	.0699	8.8446	40′	.1096	9.0398	40′	.1569	9.1956
50′	.0707	8.8492	50′	.1105	9.0434	50′	.1579	9.1985
31° 0′	.0714	8.8538	**39° 0′**	.1114	9.0470	**47° 0′**	.1590	9.2014
10′	.0722	8.8583	10′	.1123	9.0505	10′	.1601	9.2043
20′	.0729	8.8629	20′	.1133	9.0541	20′	.1611	9.2072
30′	.0737	8.8673	30′	.1142	9.0576	30′	.1622	9.2101
40′	.0744	8.8718	40′	.1151	9.0611	40′	.1633	9.2129
50′	.0752	8.8763	50′	.1160	9.0646	50′	.1644	9.2158

IV. Haversines, Natural and Logarithmic

	Nat.	Log.		Nat.	Log.		Nat.	Log.
48° 0′	.1654	9.2186	**56° 0′**	.2204	9.3432	**64° 0′**	.2808	9.4484
10′	.1665	9.2215	10′	.2216	9.3456	10′	.2821	9.4504
20′	.1676	9.2243	20′	.2228	9.3480	20′	.2834	9.4524
30′	.1687	9.2271	30′	.2240	9.3503	30′	.2847	9.4545
40′	.1698	9.2299	40′	.2252	9.3527	40′	.2861	9.4565
50′	.1709	9.2327	50′	.2265	9.3550	50′	.2874	9.4584
49° 0′	.1720	9.2355	**57° 0′**	.2277	9.3573	**65° 0′**	.2887	9.4604
10′	.1731	9.2382	10′	.2289	9.3596	10′	.2900	9.4624
20′	.1742	9.2410	20′	.2301	9.3620	20′	.2913	9.4644
30′	.1753	9.2437	30′	.2314	9.3643	30′	.2927	9.4664
40′	.1764	9.2465	40′	.2326	9.3666	40′	.2940	9.4683
50′	.1775	9.2492	50′	.2338	9.3689	50′	.2953	9.4703
50° 0′	.1786	9.2519	**58° 0′**	.2350	9.3711	**66° 0′**	.2966	9.4722
10′	.2797	9.2546	10′	.2363	9.3734	10′	.2980	9.4742
20′	.1808	9.2573	20′	.2375	9.3757	20′	.2993	9.4761
30′	.1820	9.2600	30′	.2388	9.3779	30′	.3006	9.4780
40′	.1831	9.2627	40′	.2400	9.3802	40′	.3020	9.4799
50′	.1842	9.2653	50′	.2412	9.3824	50′	.3033	9.4819
51° 0′	.1853	9.2680	**59° 0′**	.2425	9.3847	**67° 0′**	.3046	9.4838
10′	.1865	9.2706	10′	.2437	9.3869	10′	.3060	9.4857
20′	.1876	9.2732	20′	.2450	9.3891	20′	.3073	9.4876
30′	.1887	9.2759	30′	.2462	9.3913	30′	.3087	9.4895
40′	.1899	9.2785	40′	.2475	9.3935	40′	.3100	9.4914
50′	.1910	9.2811	50′	.2487	9.3957	50′	.3113	9.4932
52° 0′	.1922	9.2837	**60° 0′**	.2500	9.3979	**68° 0′**	.3127	9.4951
10′	.1933	9.2863	10′	.2513	9.4001	10′	.3140	9.4970
20′	.1945	9.2888	20′	.2525	9.4023	20′	.3154	9.4989
30′	.1956	9.2914	30′	.2538	9.4045	30′	.3167	9.5007
40′	.1968	9.2940	40′	.2551	9.4066	40′	.3181	9.5026
50′	.1979	9.2965	50′	.2563	9.4088	50′	.3195	9.5044
53° 0′	.1991	9.2991	**61° 0′**	.2576	9.4109	**69° 0′**	.3208	9.5063
10′	.2003	9.3016	10′	.2589	9.4131	10′	.3222	9.5081
20′	.2014	9.3041	20′	.2601	9.4152	20′	.3235	9.5099
30′	.2026	9.3066	30′	.2614	9.4173	30′	.3249	9.5117
40′	.2038	9.3091	40′	.2627	9.4195	40′	.3263	9.5136
50′	.2049	9.3116	50′	.2640	9.4216	50′	.3276	9.5154
54° 0′	.2061	9.3141	**62° 0′**	.2653	9.4237	**70° 0′**	.3290	9.4172
10′	.2073	9.3166	10′	.2665	9.4258	10′	.3304	9.5190
20′	.2085	9.3190	20′	.2678	9.4279	20′	.3317	6.5208
30′	.2096	9.3215	30′	.2691	9.4300	30′	.3331	9.5226
40′	.2108	9.3239	40′	.2704	9.4320	40′	.3345	9.5244
50′	.2120	9.3264	50′	.2717	9.4341	50′	.3358	9.5261
55° 0′	.2132	9.3288	**63° 0′**	.2730	9.4362	**71° 0′**	.3372	9.5279
10′	.2144	9.3312	10′	.2743	9.4382	10′	.3386	9.5297
20′	.2156	9.3336	20′	.2756	9.4403	20′	.3400	9.5314
30′	.2168	9.3361	30′	.2769	9.4423	30′	.3413	9.5332
40′	.2180	9.3384	40′	.2782	9 4444	40′	.3427	9.5349
50′	.2192	9.3408	50′	.2795	9.4464	50′	.3441	9.5367

	Nat.	Log.		Nat.	Log.		Nat.	Log.
72° 0′	.3455	9.5384	**80° 0′**	.4132	9.6161	**88° 0′**	.4826	9.6835
10′	.3469	9.5402	10′	.4146	9.6176	10′	.4840	9.6848
20′	.3483	9.5419	20′	.4160	9.6191	20′	.4855	9.6862
30′	.3496	9.5436	30′	.4175	9.6206	30′	.4869	9.6875
40′	.3510	9.5454	40′	.4189	9.6221	40′	.4884	9.6887
50′	.3524	9.5471	50′	.4203	9.6236	50′	.4898	9.6900
73° 0′	.3538	9.5488	**81° 0′**	.4218	9.6251	**89° 0′**	.4913	9.6913
10′	.3552	9.5505	10′	.4232	9.6266	10′	.4927	9.6926
20′	.3566	9.5522	20′	.4247	9.6280	20′	.4942	9.6939
30′	.3580	9.5539	30′	.4261	9.6295	30′	.4956	9.6952
40′	.3594	9.5556	40′	.4275	9.6310	40′	.4971	9.6964
50′	.3608	9.5572	50′	.4290	9.6324	50′	.4985	9.6977
74° 0′	.3622	9.5589	**82° 0′**	.4304	9.6339	**90° 0′**	.5000	9.6990
10′	.3636	9.5606	10′	.4319	9.6353	10′	.5015	9.7002
20′	.3650	9.5623	20′	.4333	9.6368	20′	.5029	9.7015
30′	.3664	9.5639	30′	.4347	9.6382	30′	.5044	9.7027
40′	.3678	9.5656	40′	.4362	9.6397	40′	.5058	9.7040
50′	.3692	9.5672	50′	.4376	9.6411	50′	.5073	9.7052
75° 0′	.3706	9.5689	**83° 0′**	.4391	9.6425	**91° 0′**	.5087	9.7065
10′	.3720	9.5705	10′	.4405	9.6440	10′	.5102	9.7077
20′	.3734	9.5722	20′	.4420	9.6454	20′	.5116	9.7090
30′	.3748	9.5738	30′	.4434	9.6468	30′	.5131	9.7102
40′	.3762	9.5754	40′	.4448	9.6482	40′	.5145	9.7114
50′	.3776	9.5771	50′	.4463	9.6496	50′	.5160	9.7126
76° 0′	.3790	9.5787	**84° 0′**	.4477	9.6510	**92° 0′**	.5174	9.7139
10′	.3805	9.5803	10′	.4492	9.6524	10′	.5189	9.7151
20′	.3819	9.5819	20′	.4506	9.6538	20′	.5204	9.7163
30′	.3833	9.5835	30′	.4521	9.6552	30′	.5218	9.7175
40′	.3847	9.5851	40′	.4535	9.6566	40′	.5233	9.7187
50′	.3861	9.5867	50′	.4550	9.6580	50′	.5247	9.7199
77° 0′	.3875	9.5883	**85° 0′**	.4564	9.6594	**93° 0′**	.5262	9.7211
10′	.3889	9.5899	10′	.4579	9.6607	10′	.5276	9.7223
20′	.3904	9.5915	20′	.4593	9.6621	20′	.5291	9.7235
30′	.3918	9.5930	30′	.4608	9.6635	30′	.5305	9.7247
40′	.3932	9.5946	40′	.4622	9.6648	40′	.5320	9.7259
50′	.3946	9.5962	50′	.4637	9.6662	50′	.5334	9.7271
78° 0′	.3960	9.5977	**86° 0′**	.4651	9.6676	**94° 0′**	.5349	9.7283
10′	.3975	9.5993	10′	.4666	9.6689	10′	.5363	9.7294
20′	.3989	9.6009	20′	.4680	9.6703	20′	.5378	9.7306
30′	.4003	9.6024	30′	.4695	9.6716	30′	.5392	9.7318
40′	.4017	9.6039	40′	.4709	9.6730	40′	.5407	9.7329
50′	.4032	9.6055	50′	.4724	9.6743	50′	.5421	9.7341
79° 0′	.4046	9.6070	**87° 0′**	.4738	9.6756	**95° 0′**	.5436	9.7353
10′	.4060	9.6085	10′	.4753	9.6770	10′	.5450	9.7364
20′	.4075	9.6101	20′	.4767	9.6783	20′	.5465	9.7376
30′	.4089	9.6116	30′	.4782	9.6796	30′	.5479	9.7387
40′	.4103	9.6131	40′	.4796	9.6809	40′	.5494	9.7399
50′	.4117	9.6146	50′	.4811	9.6822	50′	.5508	9.7410

	Nat.	Log.		Nat.	Log.		Nat.	Log.
96° 0′	.5523	9.7421	**104° 0′**	.6210	9.7931	**112° 0′**	.6873	9.8371
10′	.5537	9.7433	10′	.6224	9.7940	10′	.6887	9.8380
20′	.5552	9.7444	20′	.6238	9.7950	20′	.6900	9.8388
30′	.5566	9.7455	30′	.6252	9.7960	30′	.6913	9.8397
40′	.5580	9.7467	40′	.6266	9.7970	40′	.6927	9.8405
50′	.5595	9.7478	50′	.6280	9.7980	50′	.6940	9.8414
97° 0′	.5609	9.7489	**105° 0′**	.6294	9.7989	**113° 0′**	.6954	9.8422
10′	.5624	9.7500	10′	.6308	9.7999	10′	.6967	9.8430
20′	.5638	9.7511	20′	.6322	9.8009	20′	.6980	9.8439
30′	.5653	9.7522	30′	.6336	9.8018	30′	.6994	9.8447
40′	.5667	9.7534	40′	.6350	9.8028	40′	.7007	9.8455
50′	.5681	9.7545	50′	.6364	9.8037	50′	.7020	9.8464
98° 0′	.5696	9.7556	**106° 0′**	.6378	9.8047	**114° 0′**	.7034	9.8472
10′	.5710	9.7567	10′	.6392	9.8056	10′	.7047	9.8480
20′	.5725	9.7577	20′	.6406	9.8066	20′	.7060	9.8488
30′	.5739	9.7588	30′	.6420	9.8075	30′	.7073	9.8496
40′	.5753	9.7599	40′	.6434	9.8085	40′	.7087	9.8504
50′	.5768	9.7610	50′	.6448	9.8094	50′	.7100	9.8513
99° 0′	.5782	9.7621	**107° 0′**	.6462	9.8104	**115° 0′**	.7113	9.8521
10′	.5797	9.7632	10′	.6476	9.8113	10′	.7126	9.8529
20′	.5811	9.7642	20′	.6490	9.8122	20′	.7139	9.8537
30′	.5825	9.7653	30′	.6504	9.8131	30′	.7153	9.8545
40′	.5840	9.7664	40′	.6517	9.8141	40′	.7166	9.8553
50′	.5854	9.7674	50′	.6531	9.8150	50′	.7179	9.8561
100° 0′	.5868	9.7685	**108° 0′**	.6545	9.8159	**116° 0′**	.7192	9.8568
10′	.5883	9.7696	10′	.6559	9.8168	10′	.7205	9.8576
20′	.5897	9.7706	20′	.6573	9.8177	20′	.7218	9.8584
30′	.5911	9.7717	30′	.6587	9.8187	30′	.7231	9.8592
40′	.5925	9.7727	40′	.6600	9.8196	40′	.7244	9.8600
50′	.5940	9.7738	50′	.6614	9.8205	50′	.7257	9.8608
101° 0′	.5954	9.7748	**109° 0′**	.6628	9.8214	**117° 0′**	.7270	9.8615
10′	.5968	9.7759	10′	.6642	9.8223	10′	.7283	9.8623
20′	.5983	9.7769	20′	.6655	9.8232	20′	.7296	9.8631
30′	.5997	9.7779	30′	.6669	9.8241	30′	.7309	9.8638
40′	.6011	9.7790	40′	.6683	9.8250	40′	.7322	9.8646
50′	.6025	9.7800	50′	.6696	9.8258	50′	.7335	9.8654
102° 0′	.6040	9.7810	**110° 0′**	.6710	9.8267	**118° 0′**	.7347	9.8661
10′	.6054	9.7820	10′	.6724	9.8276	10′	.7360	9.8669
20′	.6068	9.7830	20′	.6737	9.8285	20′	.7373	9.8676
30′	.6082	9.7841	30′	.6751	9.8294	30′	.7386	9.8684
40′	.6096	9.7851	40′	.6765	9.8302	40′	.7399	9.8691
50′	.6111	9.7861	50′	.6778	9.8311	50′	.7411	9.8699
103° 0′	.6125	9.7871	**111° 0′**	.6792	9.8320	**119° 0′**	.7424	9.8706
10′	.6139	9.7881	10′	.6805	9.8329	10′	.7437	9.8714
20′	.6153	9.7891	20′	.6819	9.8337	20′	.7449	9.8721
30′	.6167	9.7901	30′	.6833	9.8346	30′	.7462	9.8729
40′	.6181	9.7911	40′	.6846	9.8354	40′	.7475	9.8736
50′	.6195	9.7921	50′	.6860	9.8363	50′	.7487	9.8743

IV. Haversines, Natural and Logarithmic

	Nat.	Log.		Nat.	Log.		Nat.	Log.
120° 0′	.7500	9.8751	**128° 0′**	.8078	9.9073	**136° 0′**	.8597	9.9343
10′	.7513	9.8758	10′	.8090	9.9079	10′	.8607	9.9348
20′	.7525	9.8765	20′	.8101	9.9085	20′	.8617	9.9353
30′	.7538	9.8772	30′	.8113	9.9092	30′	.8627	9.9359
40′	.7550	9.8780	40′	.8124	9.9098	40′	.8637	9.9364
50′	.7563	9.8787	50′	.8135	9.9104	50′	.8647	9.9369
121° 0′	.7575	9.8794	**129° 0′**	.8147	9.9110	**137° 0′**	.8657	9.9374
10′	.7588	9.8801	10′	.8158	9.9116	10′	.8667	9.9379
20′	.7600	9.8808	20′	.8169	9.9122	20′	.8677	9.9383
30′	.7612	9.8815	30′	.8180	9.9128	30′	.8686	9.9388
40′	.7625	9.8822	40′	.8192	9.9134	40′	.8696	9.9393
50′	.7637	9.8829	50′	.8203	9.9140	50′	.8706	9.9398
122° 0′	.7650	9.8836	**130° 0′**	.8214	9.9146	**138° 0′**	.8716	9.9403
10′	.7662	9.8843	10′	.8225	9.9151	10′	.8725	9.9408
20′	.7674	9.8850	20′	.8236	9.9157	20′	.8735	9.9413
30′	.7686	9.8857	30′	.8247	9.9163	30′	.8745	9.9417
40′	.7699	9.8864	40′	.8258	9.9169	40′	.8754	9.9422
50′	.7711	9.8871	50′	.8269	9.9175	50′	.8764	9.9427
123° 0′	.7723	9.8878	**131 °0′**	.8280	9.9180	**139° 0′**	.8774	9.9432
10′	.7735	9.8885	10′	.8291	9.9186	10′	.8783	9.9436
20′	.7748	9.8892	20′	.8302	9.9192	20′	.8793	9.9441
30′	.7760	9.8898	30′	.8313	9.9198	30′	.8802	9.9446
40′	.7772	9.8905	40′	.8324	9.9203	40′	.8811	9.9450
50′	.7784	9.8912	50′	.8335	9.9209	50′	.8821	9.9455
124° 0′	.7796	9.8919	**132° 0′**	.8346	9.9215	**140° 0′**	.8830	9.9460
10′	.7808	9.8925	10′	.8356	9.9220	10′	.8840	9.9464
20′	.7820	9.8932	20′	.8367	9.9226	20′	.8849	9.9469
30′	.7832	9.8939	30′	.8378	9.9231	30′	.8858	9.9473
40′	.7844	9.8945	40′	.8389	9.9237	40′	.8867	9.9478
50′	.7856	9.8952	50′	.8399	9.9242	50′	.8877	9.9482
125° 0′	.7868	9.8959	**133° 0′**	.8410	9.9248	**141° 0′**	.8886	9.9487
10′	.7880	9.8965	10′	.8421	9.9253	10′	.8895	9.9491
20′	.7892	9.8972	20′	.8431	9.9259	20′	.8904	9.9496
30′	.7904	9.8978	30′	.8442	9.9264	30′	.8913	9.9500
40′	.7915	9.8985	40′	.8452	9.9270	40′	.8922	9.9505
50′	.7927	9.8991	50′	.8463	9.9275	50′	.8931	9.9509
126° 0′	.7939	9.8998	**134° 0′**	.8473	9.9281	**142° 0′**	.8940	9.9513
10′	.7951	9.9004	10′	.8484	9.9286	10′	.8949	9.9518
20′	.7962	9.9010	20′	.8494	9.9291	20′	.8958	9.9522
30′	.7974	9.9017	30′	.8505	9.9297	30′	.8967	9.9526
40′	.7986	9.9023	40′	.8515	9.9302	40′	.8976	9.9531
50′	.7997	9.9030	50′	.8525	9.9307	50′	.8984	9.9535
127° 0′	.8009	9.9036	**135° 0′**	.8536	9.9312	**143° 0′**	.8993	9.9539
10′	.8021	9.9042	10′	.8546	9.9318	10′	.9002	9.9543
20′	.8032	9.9048	20′	.8556	9.9323	20′	.9011	9.9548
30′	.8044	9.9055	30′	.8566	9.9328	30′	.9019	9.9552
40′	.8055	9.9061	40′	.8576	9.9333	40′	.9028	9.9556
50′	.8067	9.9067	50′	.8587	9.9338	50′	.9037	9.9560

	Nat.	Log.		Nat.	Log.		Nat.	Log.
144° 0′	.9045	9.9564	**152° 0′**	.9415	9.9738	**160° 0′**	.9698	9.9867
10′	.9054	9.9568	10′	.9422	9.9741	10′	.9703	9.9869
20′	.9062	9.9572	20′	.9428	9.9744	20′	.9708	9.9871
30′	.9071	9.9576	30′	.9435	9.9747	30′	.9713	9.9874
40′	.9079	9.9580	40′	.9442	9.9751	40′	.9718	9.9876
50′	.9087	9.9584	50′	.9448	9.9754	50′	.9723	9.9878
145° 0′	.9096	9.9588	**153° 0′**	.9455	9.9757	**161° 0′**	.9728	9.9880
10′	.9104	9.9592	10′	.9462	9.9760	10′	.9732	9.9882
20′	.9112	9.9596	20′	.9468	9.9763	20′	.9737	9.9884
30′	.9121	9.9600	30′	.9475	9.9766	30′	.9742	9.9886
40′	.9129	9.9604	40′	.9481	9.9769	40′	.9746	9.9888
50′	.9137	9.9608	50′	.9488	9.9772	50′	.9751	9.9890
146° 0′	.9145	9.9612	**154° 0′**	.9494	9.9774	**162° 0′**	.9755	9.9892
10′	.9153	9.9616	10′	.9500	9.9777	10′	.9760	9.9894
20′	.9161	9.9620	20′	.9507	9.9780	20′	.9764	9.9896
30′	.9169	9.9623	30′	.9513	9.9783	30′	.9769	9.9898
40′	.9177	9.9627	40′	.9519	9.9786	40′	.9773	9.9900
50′	.9185	9.9631	50′	.9525	9.9789	50′	.9777	9.9902
147° 0′	.9193	9.9635	**155° 0′**	.9532	9.9792	**163° 0′**	.9782	9.9904
10′	.9201	9.9638	10′	.9538	9.9794	10′	.9786	9.9906
20′	.9209	9.9642	20′	.9544	9.9797	20′	.9790	9.9908
30′	.9217	9.9646	30′	.9550	9.9800	30′	.9794	9.9910
40′	.9225	9.9650	40′	.9556	9.9803	40′	.9798	9.9911
50′	.9233	9.9653	50′	.9562	9.9805	50′	.9802	9.9913
148° 0′	.9240	9.9657	**156° 0′**	.9568	9.9808	**164° 0′**	.9806	9.9915
10′	.9248	9.9660	10′	.9574	9.9811	10′	.9810	9.9917
20′	.9256	9.9664	20′	.9579	9.9813	20′	.9814	9.9919
30′	.9263	9.9668	30′	.9585	9.9816	30′	.9818	9.9920
40′	.9271	9.9671	40′	.9591	9.9819	40′	.9822	9.9922
50′	.9278	9.9675	50′	.9597	9.9821	50′	.9826	9.9924
149° 0′	.9286	9.9678	**157° 0′**	.9603	9.9824	**165° 0′**	.9830	9.9925
10′	.9293	9.9682	10′	.9608	9.9826	10′	.9833	9.9927
20′	.9301	9.9685	20′	.9614	9.9829	20′	.9837	9.9929
30′	.9308	9.9689	30′	.9619	9.9831	30′	.9841	9.9930
40′	.9316	9.9692	40′	.9625	9.9834	40′	.9844	9.9932
50′	.9323	9.9695	50′	.9630	9.9836	50′	.9848	9.9933
150° 0′	.9330	9.9699	**158° 0′**	.9636	9.9839	**166° 0′**	.9851	9.9935
10′	.9337	9.9702	10′	.9641	9.9841	10′	.9855	9.9937
20′	.9345	9.9706	20′	.9647	9.9844	20′	.9858	9.9938
30′	.9352	9.9709	30′	.9652	9.9846	30′	.9862	9.9940
40′	.9359	9.9712	40′	.9657	9.9849	40′	.9865	9.9941
50′	.9366	9.9716	50′	.9663	9.9851	50′	.9869	9.9943
151° 0′	.9373	9.9719	**159° 0′**	.9668	9.9853	**167° 0′**	.9872	9.9944
10′	.9380	9.9722	10′	.9673	9.9856	10′	.9875	9.9945
20′	.9387	9.9725	20′	.9678	9.9858	20′	.9878	9.9947
30′	.9394	9.9729	30′	.9683	9.9860	30′	.9881	9.9948
40′	.9401	9.9732	40′	.9688	9.9863	40′	.9885	9.9950
50′	.9408	9.9735	50′	.9693	9.9865	50′	.9888	9.9951

IV. Haversines, Natural and Logarithmic

	Nat.	Log.		Nat.	Log.		Nat.	Log.
168° 0′	.9891	9.9952	**172° 0′**	.9951	9.9979	**176° 0′**	.9988	9.9995
10′	.9894	9.9954	10′	.9953	9.9980	10′	.9989	9.9995
20′	.9897	9.9955	20′	.9955	9.9981	20′	.9990	9.9996
30′	.9900	9.9956	30′	.9957	9.9981	30′	.9991	9.9996
40′	.9903	9.9957	40′	.9959	9.9982	40′	.9992	9.9996
50′	.9905	9.9959	50′	.9961	9.9983	50′	.9992	9.9997
169° 0′	.9908	9.9960	**173° 0′**	.9963	9.9984	**177° 0′**	.9993	9.9997
10′	.9911	9.9961	10′	.9964	9.9984	10′	.9994	9.9997
20′	.9914	9.9962	20′	.9966	9.9985	20′	.9995	9.9998
30′	.9916	9.9963	30′	.9968	9.9986	30′	.9995	9.9998
40′	.9919	9.9965	40′	.9969	9.9987	40′	.9996	9.9998
50′	.9921	9.9966	50′	.9971	9.9987	50′	.9996	9.9998
170° 0′	.9924	9.9967	**174° 0′**	.9973	9.9988	**178° 0′**	.9997	9.9999
10′	.9927	9.9968	10′	.9974	9.9989	10′	.9997	9.9999
20′	.9929	9.9969	20′	.9976	9.9989	20′	.9998	9.9999
30′	.9931	9.9970	30′	.9977	9.9990	30′	.9998	9.9999
40′	.9934	9.9971	40′	.9978	9.9991	40′	.9999	9.9999
50′	.9936	9.9972	50′	.9980	9.9991	50′	.9999	0.0000
171° 0′	.9938	9.9973	**175° 0′**	.9981	9.9992	**179° 0′**	.9999	0.0000
10′	.9941	9.9974	10′	.9982	9.9992	10′	.9999	0.0000
20′	.9943	9.9975	20′	.9983	9.9993	20′	1.0000	0.0000
30′	.9945	9.9976	30′	.9985	9.9993	30′	1.0000	0.0000
40′	.9947	9.9977	40′	.9986	9.9994	40′	1.0000	0.0000
50′	.9949	9.9978	50′	.9987	9.9994	50′	1.0000	0.0000

V. Degrees to Radians and v.v.

n	n degrees into radians	n minutes into radians	n seconds into radians	n	n radians into degree measure
0	0.00000	0.00000	0.00000		
1	0.01745	0.00029	0.00000	0.00001	0° 0′ 02″
2	0.03491	0.00058	0.00001	0.00002	0 0 04
3	0.05236	0.00087	0.00001	0.00003	0 0 06
4	0.06981	0.00116	0.00002	0.00004	0 0 08
5	0.08727	0.00145	0.00002	0.00005	0° 0′ 10″
6	0.10472	0.00175	0.00003	0.00006	0 0 12
7	0.12217	0.00204	0.00003	0.00007	0 0 14
8	0.13963	0.00233	0.00004	0.00008	0 0 17
9	0.15708	0.00262	0.00004	0.00009	0 0 19
10	0.17453	0.00291	0.00005		
11	0.19199	0.00320	0.00005	0.0001	0° 0′ 21″
12	0.20944	0.00349	0.00006	0.0002	0 0 41″
13	0.22689	0.00378	0.00006	0.0003	0 1 02
14	0.24435	0.00407	0.00007	0.0004	0 1 23
15	0.26180	0.00436	0.00007	0.0005	0° 1′ 43″
16	0.27925	0.00465	0.00008	0.0006	0 2 04
17	0.29671	0.00495	0.00008	0.0007	0 2 24
18	0.31416	0.00524	0.00009	0.0008	0 2 45
19	0.33161	0.00553	0.00009	0.0009	0 3 06
20	0.34907	0.00582	0.00010		
21	0.36652	0.00611	0.00010	0.001	0° 03′ 26″
22	0.38397	0.00640	0.00011	0.002	0 06 53
23	0.40143	0.00669	0.00011	0.003	0 10 19
24	0.41888	0.00698	0.00012	0.004	0 13 45
25	0.43633	0.00727	0.00012	0.005	0° 17′ 11″
26	0.45379	0.00756	0.00013	0.006	0 20 38
27	0.47124	0.00785	0.00013	0.007	0 24 04
28	0.48869	0.00814	0.00014	0.008	0 27 30
29	0.50615	0.00844	0.00014	0.009	0 30 56
30	0.52360	0.00873	0.00015		
31	0.54105	0.00902	0.00015	0.01	0° 34′ 23″
32	0.55851	0.00931	0.00016	0.02	1 08 45
33	0.57596	0.00960	0.00016	0.03	1 43 08
34	0.59341	0.00989	0.00016	0.04	2 17 31
35	0.61087	0.01018	0.00017	0.05	2° 51′ 53″
36	0.62832	0.01047	0.00017	0.06	3 26 16
37	0.64577	0.01076	0.00018	0.07	4 00 39
38	0.66323	0.01105	0.00018	0.08	4 35 01
39	0.68068	0.01134	0.00019	0.09	5 09 24
40	0.69813	0.01164	0.00019		
41	0.71558	0.01193	0.00020	0.1	5° 43 46″
42	0.73304	0.01222	0.00020	0.2	11 27 33
43	0.75049	0.01251	0.00021	0.3	17 11 19
44	0.76794	0.01280	0.00021	0.4	22 55 6

V. Degrees to Radians and v.v.

n	n degrees into radians	n minutes into radians	n seconds into radians	n	n radians into degree measure
45	0.78540	0.01309	0.00022	0.5	28° 38′ 52″
46	0.80285	0.01338	0.00022	0.6	34 22 39
47	0.82030	0.01367	0.00023	0.7	40 06 25
48	0.83776	0.01396	0.00023	0.8	45 50 12
49	0.85521	0.01425	0.00024	0.9	51 33 58
50	0.87266	0.01454	0.00024		
51	0.89012	0.01484	0.00025	1.0	57° 17′ 45″
52	0.90757	0.01513	0.00025	2.0	114 35 30
53	0.92502	0.01542	0.00026	3.0	171 53 14
54	0.94248	0.01571	0.00026	4.0	229 10 59
55	0.95993	0.01600	0.00027	5.0	286° 28′ 44″
56	0.97738	0.01629	0.00027	6.0	343 46 29
57	0.99484	0.01658	0.00028	7.0	401 04 14
58	1.01229	0.01687	0.00028	8.0	458 21 58
59	1.02974	0.01716	0.00029	9.0	515 39 43
60	1.04720	0.01745	0.00029	10.0	572° 57′ 28″

VI. Mathematical Constants

$\pi = 3.14159\ 26535\ 89793.$ $\qquad \frac{1}{\pi} = 0.31830\ 98861\ 83791.$

$\pi^2 = 9.86960\ 44010\ 89359.$ $\qquad \frac{1}{\pi^2} = 0.10132\ 11836\ 42338.$

$\pi^3 = 31.00627\ 66802\ 99820.$ $\qquad \frac{1}{\pi^3} = 0.03225\ 15344\ 33199.$

$\sqrt{\pi} = 1.77245\ 38509\ 05516.$ $\qquad \frac{1}{\sqrt{\pi}} = 0.56418\ 95835\ 47756.$

$$1 \text{ radian} = \frac{180°}{\pi} = 57°.29577\ 95131,$$
$$= \frac{10800'}{\pi} = 3437'.74677\ 07849,$$
$$= \frac{648000''}{\pi} = 206264''.80624\ 70964.$$

radians.

$1° = 0.01745\ 32925\ 19943.$

$(1°)^2 = 0.00030\ 46174\ 19787.$

$(1°)^3 = 0.00000\ 53165\ 76934.$

radians.

$1' = 0.00029\ 08882\ 08666.$

$(1')^2 = 0.00000\ 00846\ 15950.$

$(1')^3 = 0.00000\ 00000\ 24614.$

$1'' = 0.00000\ 48481\ 36811.$

$(1'')^2 = 0.00000\ 00000\ 23504.$

$\sin 1° = 0.01745\ 24064\ 37284.$

$\sin 1' = 0.00029\ 08882\ 04563.$

$\sin 1'' = 0.00000\ 48481\ 36811.$

$$e = \text{Naperian base} = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots = 2.71828\ 18284\ 59045.$$

$M = 0.43429\ 44819\ 03252;\ \log_{10} n = M \log_e n.$

$\frac{1}{M} = 2.30258\ 50929\ 94046;\ \log_e n = \frac{1}{M} \log_{10} n.$

VII. Natural Logarithms and Exponential Functions

x	$\log_e x$	e^x	e^{-x}	x	$\log_e x$	e^x	e^{-x}
0.00	$-\infty$	1.000	1.000	2.50	0.916	12.18	0.082
0.05	−2.996	1.051	0.951	2.55	0.936	12.81	0.078
0.10	−2.303	1.105	0.905	2.60	0.956	13.46	0.074
0.15	−1.897	1.162	0.861	2.65	0.975	14.15	0.071
0.20	−1.610	1.221	0.819	2.70	0.993	14.88	0.067
0.25	−1.386	1.284	0.779	2.75	1.012	15.64	0 064
0.30	−1.204	1.350	0.741	2.80	1.030	16.44	0.061
0.35	−1.050	1.419	0.705	2.85	1.047	17.29	0.058
0.40	−0.916	1.492	0.670	2.90	1.065	18.17	0.055
0.45	−0.799	1.568	0.638	2.95	1.082	19.11	0.052
0.50	−0.693	1.649	0.607	3.00	1.099	20.09	0.050
0.55	−0.598	1.733	0.577	3.05	1.115	21.12	0.047
0.60	−0.511	1.822	0.549	3.10	1.131	22.20	0.045
0.65	−0.431	1.916	0.522	3.15	1.147	23.34	0.043
0.70	−0.357	2.014	0.497	3.20	1.163	24.53	0.041
0.75	−0.288	2.117	0.472	3.25	1.179	25.79	0.039
0.80	−0.223	2.226	0.449	3.30	1.194	27.11	0.037
0.85	−0.163	2.340	0.427	3.35	1.209	28.50	0.035
0.90	−0.105	2.460	0.407	3.40	1.224	29.96	0.033
0.95	−0.051	2.586	0.387	3.45	1.238	31.50	0.032
1.00	0.000	2.718	0.368	3.50	1.253	33.12	0.030
1.05	+0.049	2.858	0.350	3.55	1.267	34.81	0.029
1.10	0.095	3.004	0.333	3.60	1.281	36.60	0.027
1.15	0.140	3.158	0.317	3.65	1.295	38.47	0.026
1.20	0.182	3.320	0.301	3.70	1.308	40.45	0.025
1.25	0.223	3.490	0.287	3.75	1.322	42.52	0.024
1.30	0.262	3.669	0.273	3.80	1.335	44.70	0.022
1.35	0.300	3.857	0.259	3.85	1.348	46.99	0.021
1.40	0.337	4.055	0.247	3.90	1.361	49.40	0.020
1.45	0.372	4.263	0.235	3.95	1.374	51.94	0.019
1.50	0.406	4.482	0.223	4.00	1.386	54.60	0.018
1.55	0.438	4.711	0.212	4.05	1.399	57.40	0.017
1.60	0.470	4.953	0.202	4.10	1.411	60.34	0.017
1.65	0.501	5.207	0.192	4.15	1.423	63.43	0.016
1.70	0.531	5.474	0.183	4.20	1.435	66.69	0.015
1.75	0.560	5.755	0.174	4.25	1.447	70.11	0.014
1.80	0.588	6.050	0.165	4.30	1.459	73.70	0.014
1.85	0.615	6.360	0.157	4.35	1.470	77.48	0.013
1.90	0.642	6.686	0.150	4.40	1.482	81.45	0.012
1.95	0.668	7.029	0.142	4.45	1.493	85.63	0.012
2.00	0.693	7.389	0.135	4.50	1.504	90.02	0.011
2.05	0.718	7.768	0.129	4.55	1.515	94.63	0.011
2.10	0.742	8.166	0.122	4.60	1.526	99.48	0.010
2.15	0.766	8.585	0.116	4.65	1.537	104.58	0.010
2.20	0.789	9.025	0.111	4.70	1.548	109.95	0.009
2.25	0.811	9.488	0.105	4.75	1.558	115.58	0 009
2.30	0.833	9.974	0.100	4.80	1.569	121.51	0.008
2.35	0.854	10.486	0.095	4.85	1.579	127.74	0.008
2.40	0.876	11.023	0.091	4.90	1.589	134.29	0.007
2.45	0.896	11.588	0.086	4.95	1.599	141.17	0.007
2.50	0.916	12.182	0.082	5.00	1.609	148.41	0.007

VIII. Squares, Cubes, Square Roots, Cube Roots

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$
1	1	1	1	1	**51**	2601	132651	7.141	3.708
2	4	8	1.414	1.260	52	2704	140608	7.211	3.733
3	9	27	1.732	1.442	53	2809	148877	7.280	3.756
4	16	64	2.000	1.587	54	2916	157464	7.348	3.780
5	25	125	2.236	1.710	55	3025	166375	7.416	3.803
6	36	216	2.449	1.817	56	3136	175616	7.483	3.826
7	49	343	2.646	1.913	57	3249	185193	7.550	3.849
8	64	512	2.828	2.000	58	3364	195112	7.616	3.871
9	81	729	3.000	2.080	59	3481	205379	7.681	3.893
10	100	1000	3.162	2.154	60	3600	216000	7.746	3.915
11	121	1331	3.317	2.224	**61**	3721	226981	7.810	3.936
12	144	1728	3.464	2.289	62	3844	238328	7.874	3.958
13	169	2197	3.606	2.351	63	3969	250047	7.937	3.979
14	196	2744	3.742	2.410	64	4096	262144	8.000	4.000
15	225	3375	3.873	2.466	65	4225	274625	8.062	4.021
16	256	4096	4.000	2.520	66	4356	287496	8.124	4.041
17	289	4913	4.123	2.571	67	4489	300763	8.185	4.062
18	324	5832	4.243	2.621	68	4624	314432	8.246	4.082
19	361	6859	4.359	2.668	69	4761	328509	8.307	4.102
20	400	8000	4.472	2.714	70	4900	343000	8.367	4.121
21	441	9261	4.583	2.759	**71**	5041	357911	8.426	4.141
22	484	10648	4.690	2.802	72	5184	373248	8.485	4.160
23	529	12167	4.796	2.844	73	5329	389017	8.544	4.179
24	576	13824	4.899	2.884	74	5476	405224	8.602	4.198
25	625	15625	5.000	2.924	75	5625	421875	8.660	4.217
26	676	17576	5.099	2.962	76	5776	438976	8.718	4.236
27	729	19683	5.196	3.000	77	5929	456533	8.775	4.254
28	784	21952	5.291	3.037	78	6084	474552	8.832	4.273
29	841	24389	5.385	3.072	79	6241	493039	8.888	4.291
30	900	27000	5.477	3.107	80	6400	512000	8.944	4.309
31	961	29791	5.568	3.141	**81**	6561	531441	9.000	4.327
32	1024	32768	5.657	3.175	82	6724	551368	9.055	4.344
33	1089	35937	5.745	3.208	83	6889	571787	9.110	4.362
34	1156	39304	5.831	3.240	84	7056	592704	9.165	4.380
35	1225	42875	5.916	3.271	85	7225	614125	9.220	4.397
36	1296	46656	6.000	3.302	86	7396	636056	9.274	4.414
37	1369	50653	6.083	3.332	87	7569	658503	9.327	4.431
38	1444	54872	6.164	3.362	88	7744	681472	9.381	4.448
39	1521	59319	6.245	3.391	89	7921	704969	9.434	4.465
40	1600	64000	6.325	3.420	90	8100	729000	9.487	4.481
41	1681	68921	6.403	3.448	**91**	8281	753571	9.539	4.498
42	1764	74088	6.481	3.476	92	8464	778688	9.592	4.514
43	1849	79507	6.557	3.503	93	8649	804357	9.644	4.531
44	1936	85184	6.633	3.530	94	8836	830584	9.695	4.547
45	2025	91125	6.708	3.557	95	9025	857375	9.747	4.563
46	2116	97336	6.782	3.583	96	9216	884736	9.798	4.579
47	2209	103823	6.856	3.609	97	9409	912673	9.849	4.595
48	2304	110592	6.928	3.634	98	9604	941192	9.899	4.610
49	2401	117649	7.000	3.659	99	9801	970299	9.950	4.626
50	2500	125000	7.071	3.684	100	10000	1000000	10.000	4.642
n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$